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Qualifying scientific work as a manuscript

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THESIS

APPROXIMATION OF SPECIAL CASES OF THE HORN HYPERGEOMETRIC FUNCTIONS H_4 BY BRANCHED CONTINUED FRACTIONS

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The thesis contains the results of the original research. The use of ideas, results, and texts of other authors is referenced with the corresponding sources
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ABSTRACT

Lutsiv I.-A. Approximation of special cases of the Horn hypergeometric functions H_4 by branched continued fractions. — Qualifying scientific work as a manuscript.

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The thesis is related to the problems of establishing recurrence relations of the Horn hypergeometric series H_4, constructing the expansions of these series and their ratios in special cases into branched continued fractions, studying the convergence of branched continued fraction expansions, establishing the domains of analytical continuation of the Horn hypergeometric functions H_4 and their ratios in special cases, establishing estimates of the convergence rate and sets of numerical stability of branched continued fraction expansions.

These problems concern rational approximations of analytic functions, one of the main sections of the analytic theory of branched continued fractions. Branched continued fractions, one of the multidimensional generalizations of continued fractions, were introduced into consideration by V. Ya. Skorobohatko in 1966 together with N. S. Droniuk, O. I. Bobyk, and B. Y. Ptashnyk. The analytical theory of branched continued fractions was developed in the works of P. I. Bodnarchuk, V. Ya. Skorobogatko, D. I. Bodnar, M. S. Siavavko, Kh. Yo. Kuchminska, M. O. Nedashkovskyi, V. Siemaszko, M. O'Donohoe, J. Murphy, B. Verdonk, A. Cuyt, T. M. Antonova, O. M. Sus, R. I. Dmytryshyn, O. S. Manzii, N. P. Hoenko, V. R. Hladun, O. E. Baran, and others.

The second chapter, the first of the main sections of the thesis, is devoted to the construction of expansions of special cases of the Horn hypergeometric series H_4 into branched continued fractions. Using transformations of double power series, new three- and four-term recurrence relations for Horn hypergeometric

functions H_4 are established. Based on these recurrence relations, in Section 2.2, expansions of the Horn hypergeometric series H_4 and their ratios in special cases

$$\frac{H_4(a,c_2;c_1,c_2;\mathbf{z})}{H_4(a+1,c_2;c_1+1,c_2;\mathbf{z})},\ \frac{H_4(a,c_2+1;c_1,c_2;\mathbf{z})}{H_4(a+1,c_2+1;c_1,c_2+1;\mathbf{z})},$$
$$\frac{H_4(a,c_2+1;c_1,c_2;\mathbf{z})}{H_4(a,c_2+2;c_1,c_2+1;\mathbf{z})}$$

into branched continued fractions

$$1-z_2-\cfrac{\frac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1}{1-z_2-\cfrac{\frac{(2c_1-a+1)(a+2)}{(c_1+1)(c_1+2)}z_1}{1-z_2-\cfrac{\frac{(2c_1-a+2)(a+3)}{(c_1+2)(c_1+3)}z_1}{1-\ddots}}},$$

$$1-\frac{c_2-a}{c_2}z_2-\cfrac{\frac{2(a+1)}{c_1}z_1}{1-z_2-\cfrac{\frac{(2c_1-a-1)(a+2)}{c_1(c_1+1)}z_1}{1-z_2-\cfrac{\frac{(2c_1-a)(a+3)}{(c_1+1)(c_1+2)}z_1}{1-\ddots}}},$$

and

$$1+\cfrac{\frac{a}{c_2(c_2+1)}z_2}{1+\left(\frac{a}{c_2+1}-1\right)z_2-\cfrac{\frac{2(a+1)}{c_1}z_1}{1-z_2-\cfrac{\frac{(2c_1-a-1)(a+2)}{c_1(c_1+1)}z_1}{1-\ddots}}},$$

respectively, are constructed.

The third chapter is devoted to establishing the domains of analytic continuation of the Horn hypergeometric functions H_4 and their ratios in special cases. These domains are the domains of convergence of their expansions into branched continued fractions. The key role here is played by the principle of correspondence between the formal double power series and the branched continued fraction. Approaches are considered in which the theorem on continuation of convergence from an already known small domain (an open connected set) to a larger one is used to establish the convergence criteria of branched continued fractions.

As a result of the research in Section 3.1, convergence criteria for branched continued fraction expansions with real coefficients were established. It was proved that Cartesian products of two planes with cuts are convergence domains of the expansions of the Horn hypergeometric functions H_4 and their ratios in special cases, and, in addition, these domains are domains of analytic continuation of these functions. In particular, it was established that the domain

$$\mathcal{D}_\tau = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ z_k \notin \left[\frac{1}{4(1+\tau)}, +\infty \right), \ k = 1, 2 \right\}$$

is the domain of analytic continuation for the functions

$$\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})} \text{ and } H_4(1, c_2; c_1, c_2; \mathbf{z}),$$

where $\tau > 0$ and depends on the parameters a and c_1, and for the ratio $H_4(a, c_2+1; c_1, c_2; \mathbf{z})/H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})$ the domain of analytic continuation is the domain

$$\mathcal{P}_\tau = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin \left[\frac{1}{8\tau}, +\infty \right), \ z_2 \notin \left[\frac{1}{4\max\{1, 1 - a/(c_2+1)\}}, +\infty \right) \right\},$$

where $\tau > 0$ and depends on the parameters a, c_1 and c_2.

In Section 3.2, convergence criteria for branched continued fractions with complex coefficients are established. It is proved that the unions of bi-disks and Cartesian products of cardioid domains and half-planes are the convergence domains of these branched continued fractions, and, in addition, these domains are also domains of analytic continuation of special cases of the Horn hypergeometric functions H_4. In particular, it was established that the domain

$\mathcal{H}_{\mu,\nu}^{\kappa,\tau}=\mathcal{H}_{\mu,\nu}\cup\mathcal{H}^{\kappa,\tau}$ is the domain of analytic continuation for the functions

$$\frac{H_4(a,c_2+1;c_1,c_2;\mathbf{z})}{H_4(a+1,c_2+1;c_1,c_2+1;\mathbf{z})},$$

where

$$\mathcal{H}_{\mu,\nu}=\left\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|<\frac{1+\cos(\arg(z_1))}{2\mu},\right.$$
$$\left.\operatorname{Re}(z_2\mathrm{e}^{-(\mathrm{i}/2)\arg(z_1)})<\frac{\nu}{2}\cos\left(\frac{\arg(z_1)}{2}\right)\right\},$$
$$\mathcal{H}^{\kappa,\tau}=\left\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|<\frac{\kappa(1-\kappa)}{2\tau},\ |z_2|<\frac{1-\kappa}{2}\right\},$$

μ is a positive number, $0<\nu<1$ and $\tau>0$ and depends on the parameters a and c_1, $0<\kappa<1$, and for the ratio $H_4(a,c_2+1;c_1,c_2;\mathbf{z})/H_4(a,c_2+2;c_1,c_2+1;\mathbf{z})$ the domain of analytic continuation is the domain $\mathcal{H}_{\mu,\nu,\upsilon}^{\kappa,\tau,\upsilon}=\mathcal{H}_{\mu,\nu,\upsilon}\cup\mathcal{H}^{\kappa,\tau,\upsilon}$, where

$$\mathcal{H}_{\mu,\nu,\upsilon}=\left\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|<\frac{1+\cos(\arg(z_1))}{2\mu},\right.$$
$$\left.\operatorname{Re}(z_2\mathrm{e}^{-(\mathrm{i}/2)\arg(z_1)})>-\frac{\nu}{2\upsilon}\cos\left(\frac{\arg(z_1)}{2}\right)\right\},$$
$$\mathcal{H}^{\kappa,\tau,\upsilon}=\left\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|<\frac{\kappa(1-\kappa)}{2\tau},\ |z_2|<\frac{1-\kappa}{2\upsilon}\right\},$$

μ is a positive number, $0<\nu<1$, $0<\kappa<1$, $\tau>0$ and $\upsilon>0$ and depends on the parameters a, c_1, and c_2.

The fourth chapter is devoted to the study of the convergence rate and numerical stability of expansions of special cases of the Horn hypergeometric series H_4 into branched continued fractions. Approaches are considered in which the formula for the difference of two approximants of a branched continued fraction is used to find estimates of the approximation errors for these expansions, and the periodic continued fraction is used to establish sets of their numerical stability.

As a result, in Section 4.1, estimates of approximation errors for branched continued fractions with real coefficients in regions (a domain which may include all, part, or none of its boundary) of the space $\mathbb{R}^2$, which are Cartesian products of two semi-axes, are found. In particular, it is proved that for each $\mathbf{z} \in \mathcal{R}_\kappa$ branched continued fraction expansions of functions $H_4(a, c_2; c_1, c_2; \mathbf{z})/H_4(a + 1, c_2; c_1 + 1, c_2; \mathbf{z})$ and $H_4(1, c_2; c_1, c_2; \mathbf{z})$ converge at least as fast as geometric series with ratio

$$\frac{\tau|z_1|}{(1 - z_2)^2 + \tau|z_1|},$$

and it is also established that the domain $\mathcal{D}_\tau \cup \mathcal{P}_\tau \cup \mathrm{Int}(\mathcal{R}_\kappa)$ is the domain of analytic continuation of these functions, where

$$\mathcal{D}_\tau = \left\{\mathbf{z} \in \mathbb{C}^2 : \ z_k \notin \left[\frac{1}{4(1+\tau)}, +\infty\right), \ k = 1, 2\right\},$$

$$\mathcal{P}_\tau = \left\{\mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin \left[\frac{1}{8\tau}, +\infty\right), \ z_2 \notin \left[\frac{1}{4}, +\infty\right)\right\},$$

$$\mathcal{R}_\kappa = \left\{\mathbf{z} \in \mathbb{R}^2 : \ z_1 \le 0, \ z_2 \le \kappa\right\}, \ 0 < \kappa < 1.$$

$\tau > 0$ and depends on the parameters a and c_1,

In Section 4.2, the concept of the numerical stability set of a branched continued fraction is defined and explicit formulas for relative errors of computations of approximants of expansions of special cases of the Horn hypergeometric series H_4 into branched continued fractions are found and bi-disk sets of numerical stability for these expansions with complex coefficients are established. In particular, it is proved that the set

$$\mathcal{D}_{\kappa,\tau,\upsilon} = \left\{\mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{\kappa(1-\kappa)}{2\tau}, \ |z_2| < \frac{1-\kappa}{2\upsilon}\right\}, \ \kappa \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right)$$

is the numerical stability set for the expansion of ratio

$$\frac{H_4(a, c_2 + 1; c_1, c_2; \mathbf{z})}{H_4(a, c_2 + 2; c_1, c_2 + 1; \mathbf{z})},$$

where $\tau > 0$ and $\upsilon > 0$ and depends on the parameters a, c_1, and c_2.

Keywords: Horn hypergeometric function H_4, branched continued fraction, double power series, analytic function, approximation, continued fraction, recurrence relation, analytic continuation, convergence, rate of convergence, set of numerical stability, roundoff error.

АНОТАЦІЯ

Луців І.-А. В. Наближення спеціальних випадків гіпергеометричних функцій Горна H_4 гіллястими ланцюговими дробами. — Кваліфікаційна наукова праця на правах рукопису.

Дисертація на здобуття ступеня доктора філософії за спеціальністю 111 Математика. — Карпатський національний університет імені Василя Стефаника, Міністерство освіти і науки України. — Карпатський національний університет імені Василя Стефаника, Міністерство освіти і науки України, Івано-Франківськ, 2026.

Дисертаційна робота пов'язана із задачами встановлення рекурентних співвідношень гіпергеометричних рядів Горна H_4, побудови розвинень цих рядів та їх відношень у спеціальних випадках у гіллясті ланцюгові дроби, дослідження збіжності гіллястих ланцюгових дробових розвинень, встановлення областей аналітичного продовження гіпергеометричних функцій Горна H_4 та їх відношень у спеціальних випадках, встановлення оцінок швидкості збіжності та множин обчислювальної стійкості гіллястих ланцюгових дробових розвинень.

Ці задачі відносяться до раціональних наближень аналітичних функцій — одного із основних розділів аналітичної теорії гіллястих ланцюгових дробів. Гіллясті ланцюгові дроби — одне із багатовимірних узагальнень неперервних дробів — введено до розгляду В. Я. Скоробогатьком у 1966 році разом з Н. С. Дронюк, О. І. Бобиком, Б. Й. Пташником. Аналітична теорія гіллястих ланцюгових дробів розвивалася у працях П. І. Боднарчука, В. Я. Скоробогатька, Д. І. Боднара, М. С. Сявавка, Х. Й. Кучмінської, М. О. Недашковського, В. Семашка, М. О'Доногое, Дж. Мерфі, Б. Вердонк, А. Кайт, Т. М. Антонової, О. М. Сусь, Р. І. Дмитришина, О. С. Манзій, Н. П. Гоєнко, В. Р. Гладуна, О. Є. Баран та ін.

Другий розділ, перший із основних розділів дисертаційної роботи, присвячено побудові розвинень спеціальних випадків гіпергеометричних

рядів Горна H_4 у гіллясті ланцюгові дроби. Використовуючи перетворення подвійних степеневих рядів, встановлено нові трьох- та чотирьох-членні рекурентні співвідношення для гіпергеометричних функцій Горна H_4. На основі цих рекурентних співвідношень у підрозділі 2.2 побудовано розвинення гіпергеометричних рядів Горна H_4 та їх відношень у спеціальних випадках

$$\frac{H_4(a,c_2;c_1,c_2;\mathbf{z})}{H_4(a+1,c_2;c_1+1,c_2;\mathbf{z})},\ \frac{H_4(a,c_2+1;c_1,c_2;\mathbf{z})}{H_4(a+1,c_2+1;c_1,c_2+1;\mathbf{z})},$$
$$\frac{H_4(a,c_2+1;c_1,c_2;\mathbf{z})}{H_4(a,c_2+2;c_1,c_2+1;\mathbf{z})}$$

відповідно у гіллясті ланцюгові дроби

$$1-z_2-\cfrac{\frac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1}{1-z_2-\cfrac{\frac{(2c_1-a+1)(a+2)}{(c_1+1)(c_1+2)}z_1}{1-z_2-\cfrac{\frac{(2c_1-a+2)(a+3)}{(c_1+2)(c_1+3)}z_1}{1-\ddots}}},$$

$$1-\frac{c_2-a}{c_2}z_2-\cfrac{\frac{2(a+1)}{c_1}z_1}{1-z_2-\cfrac{\frac{(2c_1-a-1)(a+2)}{c_1(c_1+1)}z_1}{1-z_2-\cfrac{\frac{(2c_1-a)(a+3)}{(c_1+1)(c_1+2)}z_1}{1-\ddots}}},$$

та

$$1+\cfrac{\frac{a}{c_2(c_2+1)}z_2}{1+\left(\frac{a}{c_2+1}-1\right)z_2-\cfrac{\frac{2(a+1)}{c_1}z_1}{1-z_2-\cfrac{\frac{(2c_1-a-1)(a+2)}{c_1(c_1+1)}z_1}{1-\ddots}}}$$

Третій розділ присвячений встановленню областей аналітичного продовження гіпергеометричних функцій Горна H_4 та їх відношень у спеціальних випадках. Ці області є областями збіжності їхніх розвинень у гіллясті ланцюгові дроби. Ключову роль тут відіграє принцип відповідності між формальним подвійним степеневим рядом і гіллястим ланцюговим дробом. Розглянуто підходи, у яких для встановлення ознак збіжності гіллястих ланцюгових дробів використовується теорема про продовження збіжності із уже відомої малої області до більшої.

У результаті досліджень у підрозділі 3.1 встановлено ознаки збіжності для гіллястих ланцюгових дробових розвинень з дійсними коефіцієнтами. Доведено, що декартові добутки двох площин з розрізами є областями збіжності розвинень гіпергеометричних функцій Горна H_4 та їх відношень у спеціальних випадках, і, крім цього, ці області є областями аналітичного продовження цих функцій. Зокрема, встановлено, що область

$$\mathcal{D}_\tau = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ z_k \notin \left[\frac{1}{4(1+\tau)}, +\infty \right), \ k = 1, 2 \right\},$$

де $\tau > 0$ і залежить від параметрів a та c_1, є областю аналітичного продовження для функцій

$$\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})} \text{ та } H_4(1, c_2; c_1, c_2; \mathbf{z}),$$

а для відношення $H_4(a, c_2+1; c_1, c_2; \mathbf{z})/H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})$ областю аналітичного продовження є область

$$\mathcal{P}_\tau = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin \left[\frac{1}{8\tau}, +\infty \right), \ z_2 \notin \left[\frac{1}{4\max\{1, 1 - a/(c_2+1)\}}, +\infty \right) \right\},$$

де $\tau > 0$ і залежить від параметрів a, c_1 та c_2.

У підрозділі 3.2 встановлено ознаки збіжності для гіллястих ланцюгових дробів з комплексними коефіцієнтами. Доведено, що об'єднаннями бікругів і декартових добутків кардіоїдних областей і півплощин є областями збіжності цих гіллястих ланцюгових дробів, і, крім цього, ці області є також областями аналітичного продовження спеціальних випадків гіпергеометричних функцій Горна H_4. Зокрема, встановлено, що область $\mathcal{H}_{\mu,\nu}^{\kappa,\tau} =$

$\mathcal{H}_{\mu,\nu} \cup \mathcal{H}^{\kappa,\tau}$, де

$$\mathcal{H}_{\mu,\nu} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{1 + \cos(\arg(z_1))}{2\mu}, \right.$$
$$\left. \mathrm{Re}(z_2 \mathrm{e}^{-(\mathrm{i}/2)\arg(z_1)}) < \frac{\nu}{2} \cos\left(\frac{\arg(z_1)}{2}\right) \right\},$$
$$\mathcal{H}^{\kappa,\tau} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{\kappa(1-\kappa)}{2\tau}, \ |z_2| < \frac{1-\kappa}{2} \right\},$$

μ — додатне число, $0 < \nu < 1$ та $\tau > 0$ і залежить від параметрів a та c_1, $0 < \kappa < 1$, є областю аналітичного продовження для функції

$$\frac{H_4(a, c_2 + 1; c_1, c_2; \mathbf{z})}{H_4(a + 1, c_2 + 1; c_1, c_2 + 1; \mathbf{z})},$$

а для відношення $H_4(a, c_2 + 1; c_1, c_2; \mathbf{z})/H_4(a, c_2 + 2; c_1, c_2 + 1; \mathbf{z})$ областю аналітичного продовження є область $\mathcal{H}^{\kappa,\tau,\upsilon}_{\mu,\nu,\upsilon} = \mathcal{H}_{\mu,\nu,\upsilon} \cup \mathcal{H}^{\kappa,\tau,\upsilon}$, де

$$\mathcal{H}_{\mu,\nu,\upsilon} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{1 + \cos(\arg(z_1))}{2\mu}, \right.$$
$$\left. \mathrm{Re}(z_2 \mathrm{e}^{-(\mathrm{i}/2)\arg(z_1)}) > -\frac{\nu}{2\upsilon} \cos\left(\frac{\arg(z_1)}{2}\right) \right\},$$
$$\mathcal{H}^{\kappa,\tau,\upsilon} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{\kappa(1-\kappa)}{2\tau}, \ |z_2| < \frac{1-\kappa}{2\upsilon} \right\},$$

μ — додатне число, $0 < \nu < 1$, $0 < \kappa < 1$, $\tau > 0$ та $\upsilon > 0$ і залежить від параметрів a, c_1 та c_2.

Четвертий розділ присвячений дослідженню швидкості збіжності та обчислювальній стійкості розвинень спеціальних випадків гіпергеометричних рядів Горна H_4 у гіллясті ланцюгові дроби. Розглянуто підходи, у яких для знаходження оцінок похибок наближень для цих розвинень використовується формула різниці двох підхідних дробів гіллястого ланцюгового дробу, а для встановлення множин їх обчислювальної стійкості — періодичний неперервний дріб.

У результаті у підрозділі 4.1 знайдено оцінки похибок наближень для гіллястих ланцюгових дробів з дійсними коефіцієнтами у замкнених областях простору $\mathbb{R}^2$, що є декартовими добутками двох півосей. Зокрема, доведено, що для кожного $\mathbf{z} \in \mathcal{R}_\kappa$, де

$$\mathcal{R}_\kappa = \left\{\mathbf{z} \in \mathbb{R}^2 : \ z_1 \leq 0, \ z_2 \leq \kappa\right\}, \ 0 < \kappa < 1,$$

гіллясті ланцюгові дробові розвинення функцій $H_4(a, c_2; c_1, c_2; \mathbf{z})/H_4(a + 1, c_2; c_1+1, c_2; \mathbf{z})$ та $H_4(1, c_2; c_1, c_2; \mathbf{z})$ збігаються принаймні так само швидко, як геометричний ряд з членом

$$\frac{\tau|z_1|}{(1 - z_2)^2 + \tau|z_1|},$$

де $\tau > 0$ і залежить від параметрів a та c_1, а також встановлено, що область $\mathcal{D}_\tau \cup \mathcal{P}_\tau \cup \mathrm{Int}(\mathcal{R}_\kappa)$, де

$$\mathcal{D}_\tau = \left\{\mathbf{z} \in \mathbb{C}^2 : \ z_k \notin \left[\frac{1}{4(1+\tau)}, +\infty\right), \ k = 1, 2\right\},$$
$$\mathcal{P}_\tau = \left\{\mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin \left[\frac{1}{8\tau}, +\infty\right), \ z_2 \notin \left[\frac{1}{4}, +\infty\right)\right\},$$

є областю аналітичного продовження цих функцій.

У підрозділі 4.2 означено поняття множини обчислювальної стійкості гіллястого ланцюгового дробу та знайдено явні формули відносних похибок обчислень підхідних дробів розвинень спеціальних випадків гіпергеометричних рядів Горна H_4 у гіллясті ланцюгові дроби та встановлено бікругові множини обчислювальної стійкості для цих розвинень з комплексними коефіцієнтами. Зокрема, доведено, що множина

$$\mathcal{D}_{\kappa,\tau,\upsilon} = \left\{\mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{\kappa(1-\kappa)}{2\tau}, \ |z_2| < \frac{1-\kappa}{2\upsilon}\right\}, \ \kappa \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right),$$

де $\tau > 0$ та $\upsilon > 0$ і залежить від параметрів a, c_1 та c_2, є множиною обчислювальної стійкості для розвинення відношення

$$\frac{H_4(a, c_2 + 1; c_1, c_2; \mathbf{z})}{H_4(a, c_2 + 2; c_1, c_2 + 1; \mathbf{z})}.$$

Ключові слова: гіпергеометрична функція Горна H_4, гіллястий ланцюговий дріб, подвійний степеневий ряд, аналітична функція, наближення, неперервний дріб, рекурентне співвідношення, аналітичне продовження, збіжність, швидкість збіжності, множина обчислювальної стійкості, похибка заокруглення.

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CONTENTS

INTRODUCTION

Relevance of the topic. Special functions appear naturally in almost all fields of science and technology. Many of these functions are hypergeometric functions or can be expressed in terms of them. The most famous is the Gaussian hypergeometric function, which is defined by Gaussian hypergeometric series.

In approximation theory, an important problem is how to best approximate functions by simpler functions, as well as quantifying the resulting errors. The most common approximation methods include Taylor polynomial expansions, spline interpolation, Fourier series, and rational functions.

Rational approximation owes its emergence to continued fractions, which are closely related to number theory, the moment problem, the theory of orthogonal polynomials, quadrature formulas, interpolation problems in the theory of functions of complex variable, the spectral theory of self-adjoint linear operators, etc. A significant contribution to the theory of continued fractions was made by such outstanding mathematicians as L. Euler [1–3], C. F. Gauss [4], A.-M. Legendre [5], C. G. Jacobi [6], B. Riemann [7], T .Stieltjes [8, 9], and many others.

A key role in the construction of rational approximations is played by the principle of correspondence between the approximants of a continued fraction and the formal Laurent series, which represents a given analytic function of one variable. Although the basic ideas of correspondence belong to C. F. Gauss, the general theory was described by W. B. Jones and W. J. Thron (see, [10, Chapter 5]). More about studying of the continued fraction representations of special functions, including hypergeometric functions, can be found in books [10–16].

To construct rational approximations of analytical functions of several variables, a generalization of continued fractions — branched continued fractions — is used, proposed by V. Ya. Skorobohatko in 1966 together with N. S. Droniuk, O. I. Bobyk and B. Yo. Ptashnyk [17]. The analytical theory of branched continued fractions was developed in the works P. I. Bodnarchuk and V. Ya. Skorobohatko [18], D. I. Bodnar [19], M. S. Siavavko [20], Kh. Yo. Kuchminska [21], M. O. Nedashkovskyi [22], W. Siemaszko [23, 24], H. Waadeland [25], A. Cuyt and B. Verdonk [26], J. A. Murphy and M. R. O'Donohoe [27], O. M. Sus [28], T. M. Antonova [29, 30], R. I. Dmytryshyn [31, 32], O. S. Manzii [33], N. P. Hoenko [34], V. R. Hladun [35], O. E. Baran [36], and many others.

The natural two-variable extension of the hypergeometric series is the double hypergeometric series. J. Horn in 1931 listed all convergent hypergeometric series of the second order: fourteen complete series and twenty of their confluent cases [37]. Because of their applied importance, several books (see, for instance, [38, 39]) and websites (see, for example, *functions.wolfram.com*) and a large collection of articles have been devoted to these double hypergeometric series.

First expansions of the ratios of Appell hypergeometric series F_1 into formal branched continued fractions were constructed by N. S. Droniuk in [40]. However, no explicit formulas have been found for the coefficients of the constructed branched continued fractions, nor have criteria of their convergence been established. Later, N. P. Molnar and O. S. Manzii in [41] obtained the explicit formulas for such expansions. Other formal expansions for this series can be found in [34]. T. M. Antonova and N. P. Hoenko (see, [42]) was proved that the constructed expansion into a branched continued fraction converges to a function whose expansion is, and that it provides an analytic extension of this function into some domain. Expansions of the ratios of Appell hypergeometric series F_2 into formal branched continued fractions can be found in works D. I. Bodnar [43] and O. S. Manzii [33]. T. Antonova, C. Cesarano, R. Dmytryshyn and S. Sharyn [44] and R. Dmytryshyn [45, 46] investigated a special case of a branched continued fraction expansion constructed by D. I. Bodnar in [43]. An expansion of the ratio of the Appell hypergeometric series F_3 and F_4 into formal branched continued fractions was constructed and investigated, in particular, in [47] and [48], respectively. The expansions of the Horn hypergeometric series ratios H_3, H_6, and H_7 into branched continued fractions were constructed and investigated in [49,50], [51–53], and [54–56], respectively.

Despite significant achievements in the approximation of analytic functions by branched continued fractions, this topic remains one of the most important in the analytic theory of continued and branched continued fractions and one that still has many open problems, especially in the case of two variables.

One of the central problems is to establish recurrence relations for double hypergeometric series, in particular, the Horn hypergeometric series H_4, that would provide the construction of branched continued fraction expansions. Special attention is required to construct and study the expansions of Horn hypergeometric series H_4 and their ratios in special cases into branched continued fractions.

It is well known that, compared to power series, continued fractions can have wider convergence regions and better convergence rates. Here, the region refers to a domain (an open connected set) which may include all, part, or none of its boundary. These properties are also inherent to branched continued fractions. Therefore, establishing the widest convergence regions and finding estimates of approximation errors of branched continued fraction expansions, as well as proving that these branched continued fractions converge to the functions whose expansions are, and that this provides an analytic continuation of these functions into some domains, is an important scientific problem.

An equally important problem is the quantification of the resulting errors that arise when approximating analytic functions by branched continued fractions.

Connection of work with scientific programs, plans, and topics. The study was carried out within the framework of the scientific research topics "Study of algebras generated by symmetric polynomial and rational mappings in Banach spaces" (project registration number 0123U101791) and "Analysis of the spectra of countably generated algebras of symmetric polynomials and possible applications in quantum mechanics and computer science" (project registration number 2023.03/0198) of the Department of Mathematical and Functional Analysis of the Vasyl Stefanyk Carpathian National University.

Goal and objectives of the research. The goal of the research is to establish recurrence relations of the Horn hypergeometric series H_4, construct expansions of these series and their ratios in special cases into branched continued fractions, establish convergence criteria of these expansions and estimates of approximation errors for them, domains of analytical continuation of Horn hypergeometric functions H_4 and their ratios in special cases, and establish sets of numerical stability of branched continued expansions of these functions.

The object of research is the branched continued fraction expansions of Horn hypergeometric functions H_4 and their ratios in special cases.

Subject of research is the recurrence relations of the Horn hypergeometric series H_4, convergence criteria of the branched continued fraction expansions and estimates of approximation errors for them, domains of analytical continuation of the Horn hypergeometric functions H_4 and their ratios in special cases, and sets of numerical stability of branched continued expansions of these functions.

Research problems:

— to establish new recurrence relations for Horn hypergeometric series H_4;

— to construct expansions of Horn hypergeometric series H_4 and their ratios in special cases into branched continued fractions;

— to establish convergence criteria for expansions of Horn hypergeometric functions H_4 and their ratios in special cases into branched continued fractions;

— to establish domains of analytical continuation of the Horn hypergeometric functions H_4 and their ratios in special cases;

— to establish truncation error bounds for expansions of Horn hypergeometric functions H_4 and their ratios in special cases into branched continued fractions;

— to establish sets of numerical stability for expansions of ratios of Horn hypergeometric functions H_4 in special cases into branched continued fractions.

Research methods. The work uses methods of mathematical and complex analysis, as well as the analytical theory of continued and branched continued fractions.

Scientific novelty of the results obtained. All results of the thesis are new and consist of the following:

— new three- and four-term recurrence relations for the Horn hypergeometric series H_4 are established;

— expansions of the Horn hypergeometric series H_4 and their ratios in special cases into branched continued fractions are constructed;

— convergence criteria are established for expansions of Horn hypergeometric functions H_4 and their ratios in special cases into branched continued fractions in the cases of real and complex parameters;

— the domains of analytical continuation for the Horn hypergeometric functions H_4 and their ratios in special cases into branched continued fractions in the cases of real and complex parameters are established in the space $\mathbb{C}^2$;

— truncation error bounds are established for expansions of Horn hypergeometric functions H_4 and their ratios in special cases into branched continued fractions in some regions in the space $\mathbb{R}^2$;

— sets of numerical stability are established for expansions of ratios of Horn hypergeometric functions H_4 in special cases into branched continued fractions in the space $\mathbb{C}^2$.

Practical significance of the results obtained. The thesis has theoretical value. Its results can be applied in the analytical theory of continued and branched continued fractions to construct and study the expansions of double hypergeometric series and their ratios into branched continued fractions, and can also be used to construct and study the rational approximations of analytic functions of two variables that arise in applied problems in mathematics, physics, and engineering.

Personal contribution of the author. All results of the thesis submitted for defense were obtained by the author independently. In the articles [57–66] R. Dmytryshyn and C. Cesarano own the statements of the problems and the analysis of the obtained results. In the joint works with T. Antonova and S. Shatyn [57] the author owns Theorems 1 and 3, Corollaries 1–2, and Section 4, and together with O. Bodnar [63] the author owns Theorems 1–4 and Corollaries 1–2. In the joint works with M. Dmytryshyn the author owns Theorems 2–4 and Corollaries 1–2, and Section 2 in [64], Definition 1 and Theorem 1 in [60], Theorem 1 and Corollary 1 in [59], and Theorems 1–5 in [65]. In the joint work with M. Dmytryshyn and O. Kondur [58] the author owns Theorem 1 and Section 2.

Approbation of the results of the thesis. The results of the thesis were reported and discussed at:

— Nineteenth International Conference Academician Mykhailo Kravchuk (Kyiv, Ukraine, October 11–12, 2023);

— V International Conference dedicated to the 145th anniversary of the birth of Hans Hahn (Chernivtsi, Ukraine, September 23–27, 2024);

— International Workshop on Current Trends in Analysis and Approximation Theory (Rome, Italy, July 18, 2023);

— International Online Conference "Current Trends in Abstract and Applied Analysis" (Ivano-Frankivsk, Ukraine, May 12–15, 2022);

— International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications (Rome, Italy, June 25–27, 2025);

— scientific seminar of the Department of Mathematical and Functional Analysis of Vasyl Stefanyk Carpathian National University (Ivano-Frankivsk, November 13, 2024, November 11, 2025, seminar leader: Prof. Dr. A. V. Zagorodnyuk).

Publications. The results of the thesis were published in 15 printed works, including: 10 in scientific periodicals [57–66] and 5 in proceedings of international scientific conferences [67–71]; 9 articles were published in publications indexed in the Scopus and/or Web of Science Core Collection databases [57–61, 63–66].

Structure and scope of the thesis. The thesis consists of an introduction, four chapters, conclusions, references, and appendices. The thesis has 147 pages. The references take up 11 pages and contain 93 items. The appendices take up 3 pages and contain a list of publications on the topic of the thesis and information on the approval of the thesis results.

CHAPTER 1

MAIN ASPECTS AND REFERENCE OVERVIEW

This chapter provides the necessary theoretical material on continued fractions and their two-dimensional generalizations — branched continued fractions. An overview of the results of the study of continued fraction and branched continued fraction expansions for special functions with one and two variables, respectively, is presented.

1.1. Basics of continued fractions

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers, and $\mathbb{C}$ be the set of complex numbers. We write $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Continued fractions (see, [10–12,16]). Let $\langle \{a_k\}_{k\geq1}, \{b_k\}_{k\geq0} \rangle$ be an ordered pair of sequences of complex numbers such that $a_k \neq 0$, $k \geq 1$, and $\{t_k(\xi)\}_{k\geq0}$ and $\{T_k(\xi)\}_{k\geq0}$ be the sequences of linear fractional transformations defined as follows

$$t_0(\xi) = b_0 + \xi,\ t_k(\xi) = \frac{a_k}{b_k + \xi},\ k \geq 1,$$
$$T_0(\xi) = t_0(\xi),\ T_k(\xi) = T_{k-1}(t_k(\xi)),\ k \geq 1.$$

Let $\{f_k\}_{k\geq0}$ be a sequence in $\widehat{\mathbb{C}}$, given as

$$f_k = T_k(0),\ k \geq 0.$$

The ordered pair [72, p. 474]

$$\langle \langle \{a_k\}_{k\geq1}, \{b_k\}_{k\geq0} \rangle, \{f_k\}_{k\geq0} \rangle$$

is the continued fraction denoted by the symbol

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}. \tag{1.1}$$

The numbers a_k and b_k are called kth partial numerator and partial denominator of the continued fraction, respectively. They are also called elements of (1.1). The value f_k is called the kth approximant and is denoted by the symbol

$$f_k = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_k}{b_k}}}. \tag{1.2}$$

For convenience, we will also denote a continued fraction by the symbol [73, p. 148]

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots$$

and the symbol [74]

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k}{b_k},$$

and its kth approximant by

$$f_k = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_k}{+ b_k}$$

and

$$f_k = b_0 + \mathop{\mathrm{D}}_{n=1}^{k} \frac{a_n}{b_n},$$

respectively.

We will also consider continued fractions, in which the partial numerators can be zero. In this case, we assume the following: the approximant (1.2) does not makes sense [16, p. 14] if we obtain an uncertainty of $0/0$, when it is folded from bottom to top without any reductions in intermediate operations.

A continued fraction (1.1) converges [16, p. 16], if at most a finite number of its approximants does not make sense, and the sequence $\{f_k\}_{k\geq 0}$ converges to a finite limit f, i.e.

$$f = \lim_{k\to+\infty} f_k, \; f \in \mathbb{C}.$$

In this case, f is called the value of the continued fraction (1.1). Otherwise, the continued fraction (1.1) diverges.

Let

$$d_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{c_k}{d_k} \tag{1.3}$$

be a continued fraction where $d_{k-1} \in \mathbb{C}$, $c_k \in \mathbb{C} \setminus \{0\}$, $k \geq 1$.

Two continued fractions (1.1) and (1.3) with sequences of approximants $\{f_k\}_{k \geq 0}$ and $\{g_k\}_{k \geq 0}$, respectively, are called equivalent [75], if these sequences coincide term by term, i.e.

$$f_k = g_k, \; k \geq 0.$$

In this case, we will write (see [11, p. 15] and [14, p. 5])

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k}{b_k} \equiv d_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{c_k}{d_k}. \tag{1.4}$$

The equivalence property (1.4) holds if and only if there exist nonzero constants ρ_k, $k \geq 1$, such that

$$d_0 = b_0, \; c_1 = \rho_1 a_1, \; c_{k+1} = \rho_k \rho_{k+1} a_{k+1}, \; d_k = \rho_k b_k, \; k \geq 1. \tag{1.5}$$

The transformation of a continued fraction (1.1) into (1.3), determined by the relations (1.5) for some ρ_k, $k \geq 1$, is called an equivalent transformation [75] (see also [11, p. 15–16], [10, pp. 31–36], [14, § 2], [15, § 42], [16, pp. 19–20]).

If (1.1) is a continued fraction with positive elements, then the so-called fork property holds [76, 77], i.e.

$$f_{2k-2} < f_{2k} < f_{2k+1} < f_{2k-1}, \; k \geq 1.$$

Many convergence criteria of continued fractions have been established, including periodic continued fractions [10–14, 16]. To prove our results, we will use the following theorem.

Theorem 1.1 ([10, Theorem 3.2]). *The periodic continued fraction*

$$\cfrac{a}{1 + \cfrac{a}{1 + \cfrac{a}{1 + \ddots}}}$$

converges for all non-zero complex numbers a unless $a = -1/4 + c$, where c is real and negative number. For

$$a = -\frac{1}{4} + c,\ c = |c|\mathrm{e}^{\mathrm{i}\gamma},\ -\pi < \gamma < \pi$$

the value of this periodic continued fraction is

$$\frac{-1 + 2\sqrt{|c|}\mathrm{e}^{\mathrm{i}\gamma/2}}{2}.$$

There are several algorithms for computing the nth approximant of the continued fraction (1.1), among which the most widely used is the backward recurrence algorithm (see, for example, [10, p. 352]) which uses the the following recurrence relations

$$F_k^{(n)} = b_k + \frac{a_{k+1}}{F_{k+1}^{(n)}},\ n-1 \geq k \geq 0,\ n \geq 1,$$

with initial conditions

$$F_n^{(n)} = b_n,\ n \geq 0.$$

Then

$$f_n = F_0^{(n)},\ n \geq 0.$$

The terms $F_k^{(n)}$, $0 \leq k \leq n$, are called tails of nth approximant of continued fraction (1.1) [78].

An important application of continued fractions is the representation of analytic functions by continued fractions of the form

$$b_0(z) + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k(z)}{b_k(z)}, \tag{1.6}$$

where $b_{k-1}(z)$, $a_k(z)$, $k \geq 1$, are polynomials in z or $1/z$. In this case, the so-called concept of correspondence is used.

Correspondence (see, [11, pp. 30–35], [10, pp. 147–160], [13, pp. 241–290]). The expression

$$L(z) = \sum_{k=r}^{+\infty} c_k z^k \tag{1.7}$$

is called the formal Laurent series at the point $z = 0$, where $r \in \mathbb{Z}$, $c_k \in \mathbb{C}$, $k \geq r$, and $c_r \neq 0$ or all $c_k = 0$. Let the symbol 0 denote the neutral element for the operation of addition of formal Laurent series. The set $\mathbb{L}$ of all formal Laurent series at the point $z = 0$ forms a field over $\mathbb{C}$ with respect to the operations of addition and multiplication. If $r \geq 0$, then the expression (1.7) is called the formal power series at the point $z = 0$.

For all $L(z) \in \mathbb{L}$, we define the function $\lambda : \mathbb{L} \to \mathbb{Z} \cup \{+\infty\}$ as follows:

$$\lambda(L) = \begin{cases} r \text{ if } L(z) = \sum\limits_{k=r}^{+\infty} c_k z^k, \ c_r \neq 0, \\ +\infty \text{ if } L(z) \equiv 0. \end{cases} \tag{1.8}$$

For a function $f(z)$, meromorphic at the point $z = 0$, we denote its expansion into a formal Laurent series at the origin (the one that coincides in some neighborhood with the punctured point $z = 0$) by $\Lambda(f)$, i.e. we define the mapping $\Lambda : \ f(z) \to \Lambda(f)$.

The sequence of functions $\{f_k(z)\}_{k\geq 0}$, meromorphic at the origin, is corresponding to the formal Laurent series $L(z)$ at the point $z = 0$, if

$$\lim_{k\to+\infty} \lambda(L - \Lambda(f_k)) = +\infty.$$

If the sequence $\{f_k(z)\}_{k\geq 0}$ is corresponding to the formal Laurent series $L(z)$, then the order of correspondence of the function $f_k(z)$ is defined as follows:

$$\nu_k = \lambda(L - \Lambda(f_k)).$$

In this case, by the definition of the function λ in (1.8) it follows that the formal Laurent series $L(z)$ and $\Lambda(f_k)$ converge term by term up to the term with the power $(\nu_k - 1)$ inclusive.

A continued fraction of the form (1.6) is called corresponding to the formal Laurent series $L(z)$ at the point $z = 0$, if the sequence of its approximants $\{f_k(z)\}_{k\geq 0}$ is corresponding to $L(z)$ at the point $z = 0$. A sequence of functions $\{f_k(z)\}_{k\geq 0}$ (or continued fraction (1.6)) is called corresponding at the point $z = 0$ to the function $f(z)$, meromorphic at the origin, if this sequence (or continued fraction) corresponds to the formal Laurent series $\Lambda(f)$ at the point $z = 0$.

Uniform convergence [10, p. 176]. Let $\mathcal{D}$ be some domain in $\mathbb{C}$. Here, the domain is an open connected subset of $\mathbb{C}$.

A sequence $\{f_k(z)\}_{k\geq 1}$ of functions meromorphic in the domain $\mathcal{D}$ is said to converge uniformly on a compact subset $\mathcal{K}$ of $\mathcal{D}$ if:

(i) there exists $N(\mathcal{K})$ such that $f_k(z)$ is holomorphic in the domain $\mathcal{K}$ for all $k \geq N(\mathcal{K})$, and

(ii) given $\varepsilon > 0$ there exists $N_\varepsilon > N(\mathcal{K})$ such that

$$\sup_{z\in\mathcal{K}} |f_{k+r}(z) - f_k(z)| < \varepsilon,\ k \geq N_\varepsilon,\ r \geq 0.$$

The sequence $\{f_k(z)\}_{k\geq 1}$ of functions holomorphic in the domain $\mathcal{D}$ is said to be uniformly bounded on a compact subset $\mathcal{K}$ of $\mathcal{D}$ if there exist $N(\mathcal{K})$ and $M(\mathcal{K})$ such that

$$\sup_{z\in\mathcal{K}} |f_k(z)| < M(\mathcal{K}),\ k \geq N(\mathcal{K}).$$

A continued fraction (1.6) is said to converge at $z = z_0$ if its sequence of approximants $\{f_k(z_0)\}_{k\geq 0}$ converges, and

$$\lim_{k\to+\infty} f_k(z_0)$$

is called its value.

A continued fraction (1.6) is said to converge uniformly on a compact subset $\mathcal{K}$ of $\mathcal{D}$ if its sequence of approximants $\{f_k(z)\}_{k\geq 0}$ converges uniformly on $\mathcal{K}$.

Let $\mathcal{O}$ be some region in $\mathbb{C}$. Here, the region refers to a domain which may include all, part, or none of its boundary.

If for each $z \in \mathcal{O}$ the continued fraction (1.6) converges to the finite value $f(z)$, then, for $k \geq 0$,

$$f(z) - f_k(z)$$

is called the truncation error of the kth approximant [10, Chapter 8]. For $k \geq 0$ the estimate of the form

$$|f(z) - f_k(z)| \leq C_k(z)$$

is called a priori bound (or truncation error bound), where $C_k(z) \geq 0$, $k \geq 0$, and $C_k(z) \to 0$ as $k \to +\infty$ for $z \in \mathcal{O}$.

1.2. Continued fraction representations of functions

Many special functions in mathematics, physics, and engineering are hypergeometric functions or combinations thereof (see, [10–12, 38, 79, 80]).

The hypergeometric function is defined as follows [4]

$$F(a, b; c; z) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.9}$$

where $a, b, c \in \mathbb{C}$, $c \notin \{0, -1, -2, \ldots\}$, $z \in \mathbb{C}$,

$$(\alpha)_0 = 1,\ (\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1),\ \alpha \in \mathbb{C},\ k \geq 1,$$

is the Pochhammer symbol. This function is also known as the Gaussian hypergeometric function. The power series (1.9) converges for $|z| < 1$ and diverges for $|z| > 1$.

Example 1.1 ([79, p. 556]). *The following special functions are expressed in terms of the hypergeometric functions*

$$F(1/2, 1; 3/2; z^2) = \frac{1}{2z} \ln\left(\frac{1+z}{1-z}\right)$$

and

$$\frac{F(1/2, -1/2; 1/2; z^2)}{F(1/2, 1/2; 3/2; z^2)} = \frac{z\sqrt{1-z^2}}{\arcsin(z)}.$$

Note that if $f(z)$ is a multivalued function, then we will always take its principal value.

There exist 26 three-term recurrence relations of Gauss hypergeometric functions [11, p. 294], also called contiguous relations [80, p. 94].

From the contiguous relation

$$F(a,b;c;z) = F(a,b+1;c+1;z) - \frac{a(c-b)}{c(c+1)} zF(a+1,b+1;c+2;z)$$

we obtain the continued fraction [4]

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} = 1 - \frac{u_1 z}{1} \underset{-}{} \frac{u_2 z}{1} \underset{-}{} \ldots, \; z \in \mathbb{C} \setminus [1,+\infty),$$

also called the Gauss continued fraction, where $a, b \in \mathbb{C}$, $c \notin \{0, -1, -2, \ldots\}$,

$$u_{2k-1} = \frac{(a+k-1)(c-b+k-1)}{(c+2k-2)(c+2k-1)}, \; u_{2k} = \frac{(b+k)(c-a+k)}{(c+2k-1)(c+2k)}, \; k \geq 1.$$

Setting $b = 0$ and replacing c by $c-1$, we have

$$F(a,1;c;z) = \frac{1}{1} \underset{-}{} \frac{v_1 z}{1} \underset{-}{} \frac{v_2 z}{1} \underset{-}{} \ldots, \; z \in \mathbb{C} \setminus [1,+\infty), \tag{1.10}$$

where $a, c \in \mathbb{C}$, $c \notin \{1, 0, -1, -2, \ldots\}$, and

$$v_{2k-1} = \frac{(a+k-1)(c+k-2)}{(c+2k-3)(c+2k-2)}, \; v_{2k} = \frac{k(c-a+k-1)}{(c+2k-2)(c+2k-1)}, \; k \geq 1.$$

Example 1.2 ([11, p. 297]). *From* (1.10) *it follows that*

$$F(1/2,1;3/2;z) = \frac{1}{2\sqrt{z}} \ln\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)$$

has the continued fraction

$$F(1/2,1;3/2;z) = \frac{1}{1} \underset{-}{} \frac{w_1 z}{1} \underset{-}{} \frac{w_2 z}{1} \underset{-}{} \ldots, \; z \in \mathbb{C} \setminus [1,+\infty),$$

where

$$w_k = \frac{k^2}{4k^2-1}, \; k \geq 1.$$

Confluent hypergeometric functions. The function defined by [79, p. 504]

$$M(a;b;z) = \sum_{k=0}^{+\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, \; z \in \mathbb{C},$$

is called a confluent hypergeometric function, where a, b are complex constant herewith $b \notin \{0, -1, -2, \ldots\}$. The function $M(a;b;z)$ also called the Kummer function.

For function

$$\frac{M(a,b;z)}{M(a+1;b+1;z)}$$

we obtain (see, [10, p. 206])

$$\frac{M(a,b;z)}{M(a+1;b+1;z)} = 1 + \frac{h_1 z}{1} \, \frac{}{+} \, \frac{h_2 z}{1} \, \frac{}{+} \, \ldots, \quad z \in \mathbb{C}, \tag{1.11}$$

where $a, b \in \mathbb{C}$, $b \notin \{1, 0, -1, -2, \ldots\}$, and

$$h_{2k-1} = -\frac{b-a+k-1}{(b+2k-2)(b+2k-1)}, \; h_{2k} = \frac{a+k}{(b+2k-1)(b+2k)}, \; k \geq 1.$$

From (1.11) we have the continued fraction

$$M(1;b+1;z) = \frac{1}{1} \, \frac{}{+} \, \frac{d_1 z}{1} \, \frac{}{+} \, \frac{d_2 z}{1} \, \frac{}{+} \, \ldots, \quad z \in \mathbb{C}, \tag{1.12}$$

where $b \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and

$$d_{2k-1} = -\frac{b+k-1}{(b+2k-2)(b+2k-1)}, \; h_{2k} = \frac{k}{(b+2k-1)(b+2k)}, \; k \geq 1.$$

Example 1.3 ([11, p. 323])**.** *From* (1.12) *it follows that*

$$M(1;2;z) = \frac{e^z - 1}{z}$$

has the continued fraction

$$M(1;2;z) = \frac{1}{1} \, \frac{}{+} \, \frac{f_1 z}{1} \, \frac{}{+} \, \frac{f_2 z}{1} \, \frac{}{+} \, \ldots, \quad z \in \mathbb{C},$$

where

$$f_{2k-1} = -\frac{1}{2(2k-1)}, \; f_{2k} = \frac{1}{2(2k-1)}, \; k \geq 1.$$

More examples of representing of the special functions as continued fractions can be found in books [10, Chapter 6], [11, pp. 193–400], [12, Appendix A], [81, pp. 193–212], and [82, Chapter 2].

1.3. Basics of branched continued fractions

As a generalization of continued fractions (1.1), branched continued fractions

$$\sum_{i_1=1}^{N}\cfrac{a_{i_1}}{b_{i_1}+\sum_{i_2=1}^{N}\cfrac{a_{i_1,i_2}}{b_{i_1,2}+\sum_{i_3=1}^{N}\cfrac{a_{i_1,i_2,i_3}}{b_{i_1,i_2,i_3}+\ddots}}}$$

were proposed by V. Skorobohatko in 1966 together with N. Droniuk, O. Bobyk and B. Ptashnyk [17], where N is a fixed natural number, $a_{i_1}, b_{i_1}, a_{i_1,i_2}, b_{i_1,i_2}, \ldots$ are the elements (this can be numbers, functions, etc.). The basics of the analytic theory of branched continued fractions is presented in the books [18, 19, 21].

We consider branched continued fractions with $N = 2$. Let $i(0) = 0$, $\mathcal{I}_0 = \{0\}$, and

$$\mathcal{I}_k = \{i(k) : \ i(k) = (i_1, i_2, \ldots, i_k),\ 1 \le i_r \le 2,\ 1 \le r \le k\},\ k \ge 1,$$

be the sets of multiindices. For each $l \ge 1$ the symbol $\mathbf{u}^{(l)}$ (see, [19, p. 15]) denotes a vector in $\mathbb{C}^{2^l}$ with components $u_{j(l)}$, $j(l) \in \mathcal{I}_l$; for each $l \ge 1$, $k \ge 1$, and for each multiindex $i(k) \in \mathcal{I}_k$ the symbol $\mathbf{u}^{(l)}_{i(k)}$ is a vector in $\mathbb{C}^{2^l}$ with components

$$u_{i(k),j(l)},\ i(k) \in \mathcal{I}_k,\ 1 \le j_p \le 2,\ 1 \le p \le l,\ j_0 = i_k,$$

with the following order of components:

(i) $u_{n(l)} \prec u_{m(l)}$ $(u_{i(k),n(l)} \prec u_{i(k),m(l)})$, if $n(l) \prec m(l)$;

(ii) $n(l) \prec m(l)$, if $n_1 < m_1$ or there exists index p, $1 \le p < l$, such that $n_r = m_r$, $1 \le r \le p$, and $n_{p+1} < m_{p+1}$.

Let

$$\langle \{a_{i(k)}\}_{i(k)\in\mathcal{I}_k,\ k\ge1}, \{b_{i(k)}\}_{i(k)\in\mathcal{I}_k,\ k\ge0}\rangle$$

be an ordered pair of sequences of complex numbers such that:

(i) $a_{i(k)} \neq 0$ for all $i(k) \in \mathcal{I}_k$, $k \geq 1$;

(ii) if for $k \geq 1$ there exists a multiindex $i(k) \in \mathcal{I}_k$ such that $b_{i(k)} = 0$, then $b_{i(k-1),j} \neq 0$ for $1 \leq j \leq i_{k-1}$ and $j \neq i_k$.

Let

$$\{t_{i(k)}(\boldsymbol{\xi}^{(1)}_{i(k)})\}_{i(k)\in\mathcal{I}_k,\ k\geq 0}$$

herewith $\boldsymbol{\xi}^{(1)}_0 = \boldsymbol{\xi}^{(1)}$ and

$$\{T_k(\boldsymbol{\xi}^{(k+1)})\}_{k\geq 0}$$

be the sequences of two-dimensional linear fractional transformations defined as follows

$$t_0(\boldsymbol{\xi}^{(1)}) = b_0 + \xi_1 + \xi_2,$$
$$\eta_{i(k)} = t_{i(k)}(\boldsymbol{\xi}^{(1)}_{i(k)}) = \frac{a_{i(k)}}{b_{i(k)} + \xi_{i(k),1} + \xi_{i(k),2}},\ i(k) \in \mathcal{I}_k,\ k \geq 1,$$

and

$$T_0(\boldsymbol{\xi}^{(1)}) = t_0(\boldsymbol{\xi}^{(1)}),\ T_k(\boldsymbol{\xi}^{(k+1)}) = T_{k-1}(\boldsymbol{\eta}^{(k)}),\ k \geq 1.$$

Next, let $\{f_k\}_{k\geq 0}$ be a sequence in $\widehat{\mathbb{C}}$ given as

$$f_k = T_k(\mathbf{0}^{(k+1)}),\ k \geq 0,$$

where $\mathbf{0}^{(k+1)} = (0, 0, \ldots, 0)$ is a vector in $\mathbb{C}^{2^{k+1}}$.

The ordered pair

$$\langle\langle\{a_{i(k)}\}_{i(k)\in\mathcal{I}_k,k\geq 1}, \{b_{i(k)}\}_{i(k)\in\mathcal{I}_k,k\geq 0}\rangle, \{f_k\}_{k\geq 0}\rangle$$

is the branched continued fraction denoted by symbol

$$b_0 + \sum_{i_1=1}^{2} \cfrac{a_{i(1)}}{b_{i(1)} + \sum\limits_{i_2=1}^{2} \cfrac{a_{i(2)}}{b_{i(2)} + \sum\limits_{i_3=1}^{2} \cfrac{a_{i(3)}}{b_{i(3)} + \ddots}}}. \tag{1.13}$$

The numbers $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, and $b_{i(k)}$, $i(k) \in \mathcal{I}_k$, are called the kth partial numerators and partial denominators, respectively, and are also called the elements. The value

$$f_k = b_0 + \sum_{i_1=1}^{2} \cfrac{a_{i(1)}}{b_{i(1)} + \sum\limits_{i_2=1}^{2} \cfrac{a_{i(2)}}{b_{i(2)} + \ddots + \sum\limits_{i_k=1}^{2} \cfrac{a_{i(k)}}{b_{i(k)}}}} \tag{1.14}$$

is called the kth approximant of the branched continued fraction (1.13).

To write a branched continued fraction, we will also use the following symbols:

$$b_0 + \sum_{i_1=1}^{2} \frac{a_{i(1)}}{b_{i(1)}} {+} \sum_{i_2=1}^{2} \frac{a_{i(2)}}{b_{i(2)}} {+} \sum_{i_3=1}^{2} \frac{a_{i(3)}}{b_{i(3)}} {+} \cdots$$

and

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{+\infty} \sum_{i_k=1}^{2} \frac{a_{i(k)}}{b_{i(k)}},$$

and for kth approximant,

$$f_k = b_0 + \sum_{i_1=1}^{2} \frac{a_{i(1)}}{b_{i(1)}} {+} \sum_{i_2=1}^{2} \frac{a_{i(2)}}{b_{i(2)}} {+} \cdots {+} \sum_{i_k=1}^{2} \frac{a_{i(k)}}{b_{i(k)}}$$

and

$$f_k = b_0 + \mathop{\mathrm{D}}_{n=1}^{k} \sum_{i_n=1}^{2} \frac{a_{i(n)}}{b_{i(n)}},$$

respectively.

As in the one-dimensional case, we will also consider branched continued fractions, in which the partial numerators and, under certain conditions, the partial denominators can be zero. We assume that for all $a \in \mathbb{C}$ and $b \in \mathbb{C}$

$$\frac{a}{0} + \frac{b}{0} = \frac{0}{0},$$

and that the approximant (1.14) makes sense if we do not obtain the uncertainty of 0/0, when it is folded from bottom to top without any reductions in intermediate operations [19, pp. 20–27].

The branched continued fraction (1.13) converges [19, p. 46], if at most a finite number of its approximants does not make sense, and the sequence $\{f_k\}_{k\geq 0}$ converges to a finite limit f, i.e.

$$f=\lim_{k\to+\infty} f_k,\ f\in\mathbb{C}.$$

In this case, f is called the value of the branched continued fraction (1.13). Otherwise, the branched continued fraction (1.13) diverges.

If (1.13) is a branched continued fraction with positive elements, then the so-called fork property holds [19, p. 29], i.e.

$$f_{2k-2}<f_{2k}<f_{2k+1}<f_{2k-1},\ k\geq 1.$$

Formula of difference of two approximants [83]. Let $F^{(n)}_{i(k)}(\mathbf{z})$, $i(k)\in\mathcal{I}_k$, $1\leq k\leq n$, $n\geq 1$, denote the so-called tails of branched continued fraction (1.13), that is

$$F^{(n)}_{i(n)}=b_{i(n)},\ i(n)\in\mathcal{I}_n,\ n\geq 1,$$

and

$$F^{(n)}_{i(k)}=b_{i(k)}+\sum_{i_{k+1}=1}^{2}\cfrac{a_{i(k+1)}}{b_{i(k+1)}+\sum\limits_{i_{k+2}=1}^{2}\cfrac{a_{i(k+2)}}{b_{i(k+2)}+\ddots+\sum\limits_{i_n=1}^{2}\cfrac{a_{i(n)}}{b_{i(n)}}}},$$

where $i(k)\in\mathcal{I}_k$, $1\leq k\leq n-1$, $n\geq 2$. Then

$$F^{(n)}_{i(k)}=b_{i(k)}+\sum_{i_{k+1}=1}^{2}\frac{a_{i(k+1)}}{F^{(n)}_{i(k+1)}},\ i(k)\in\mathcal{I}_k,\ 1\leq k\leq n-1,\ n\geq 2,$$

and

$$f_n=b_0+\sum_{i_1=1}^{2}\frac{a_{i(1)}}{F^{(n)}_{i(1)}},\ n\geq 1.$$

If $F_{i(k)}^{(n)} \neq 0$, $i(k) \in \mathcal{I}_k$, $1 \leq k \leq n$, $n \geq 1$, then for $m > n \geq 1$ the formula holds

$$f_m - f_n = (-1)^n \sum_{i_1=1}^{2} \frac{a_{i(1)}}{F_{i(1)}^{(m)} F_{i(1)}^{(n)}} \cdots \sum_{i_n=1}^{2} \frac{a_{i(n)}}{F_{i(n)}^{(m)} F_{i(n)}^{(n)}} \sum_{i_{n+1}=1}^{2} \frac{a_{i(n+1)}}{F_{i(n+1)}^{(m)}}. \tag{1.15}$$

Let

$$d_0 + \mathop{\mathrm{D}}_{k=1}^{+\infty} \sum_{i_k=1}^{2} \frac{c_{i(k)}}{d_{i(k)}} \tag{1.16}$$

be a branched continued fraction with sequences of approximants $\{g_k\}_{k \geq 0}$.

A branched continued fraction (1.16) is called the majorant [19, p. 51] of a branched continued fraction (1.13) if there exist $n_0 > 0$ and $M > 0$ such that

$$|f_{n+k} - f_n| \leq M |g_{n+k} - g_n|, \ n \geq n_0, \ k \geq 0.$$

Let $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ and $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. An important application of branched continued fractions is, in particular, to represent special functions expressed by formal double power series at $\mathbf{z} = \mathbf{0}$

$$L(\mathbf{z}) = \sum_{p,q=0}^{+\infty} a_{p,q} z_1^p z_2^q$$

as branched continued fractions

$$b_0(\mathbf{z}) + \mathop{\mathrm{D}}_{k=1}^{+\infty} \sum_{i_k=1}^{2} \frac{a_{i(k)}(\mathbf{z})}{b_{i(k)}(\mathbf{z})}, \tag{1.17}$$

where $a_{p,q} \in \mathbb{C}$, $p \geq 0$, $q \geq 0$, $b_0(\mathbf{z})$, $a_{i(k)}(\mathbf{z})$, $b_{i(k)}(\mathbf{z})$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are polynomials in $\mathbf{z}$. The key points here are the correspondence principle and the fork property.

Correspondence (see, [57] and [11, pp. 30–35]). Let $\mathbb{L}$ be the set of formal double power series $L(\mathbf{z})$ at the $\mathbf{z} = \mathbf{0}$, $f(\mathbf{z})$ be function holomorphic in a neighbourhood of the origin $\mathbf{z} = \mathbf{0}$, and let the mapping $\Lambda : f(\mathbf{z}) \to \Lambda(f)$ associate with $f(\mathbf{z})$ its Taylor expansion in a neighbourhood of the origin.

A sequence $\{f_k(\mathbf{z})\}_{k\geq 0}$ of functions holomorphic at the origin corresponds to a formal double power series $L(\mathbf{z})$ at $\mathbf{z}=\mathbf{0}$ if

$$\lim_{k\to+\infty}\lambda(L-\Lambda(f_k))=+\infty,$$

where $\lambda:\mathbb{L}\to\mathbb{Z}_{\geq 0}\cup\{+\infty\}$ is the function defined as follows:

$$\lambda(L)=\begin{cases} k \text{ if } L(\mathbf{z})\not\equiv 0,\\ +\infty \text{ if } L(\mathbf{z})\equiv 0,\end{cases}$$

where k is the smallest degree of homogeneous terms for which $a_{p,q}\neq 0$, that is $k=p+q$.

If $\{f_k(\mathbf{z})\}_{k\geq 0}$ corresponds at $\mathbf{z}=\mathbf{0}$ to a formal double power series $L(\mathbf{z})$, then the order of correspondence of $f_k(\mathbf{z})$ is defined to be

$$\nu_k=\lambda(L-\Lambda(f_k)).$$

By the definition of λ, the series $L(\mathbf{z})$ and $\Lambda(f_k)$ agree for all homogeneous terms up to and including degree (ν_k-1).

A branched continued fraction (1.17), whose elements are polynomials in $\mathbb{C}^2$, corresponds to a formal double power series $L(\mathbf{z})$ at $\mathbf{z}=\mathbf{0}$ if its sequence of approximants corresponds to $L(\mathbf{z})$.

To prove our results, we will use the following two theorems.

Theorem 1.2 (Weierstrass' Theorem [84, p. 23]). *Let a sequence $\{f_k(\mathbf{z})\}_{k\geq 1}$ of holomorphic functions in a domain $\mathcal{D}$ of $\mathbb{C}^2$ converges to a function $f(\mathbf{z})$ uniformly on each compact subset in $\mathcal{D}$, then $f(\mathbf{z})$ is holomorphic in the domain $\mathcal{D}$, and for any $p\geq 0$, $q\geq 0$,*

$$\frac{\partial^{p+q}f_k(\mathbf{z})}{\partial z_1^p\partial z_2^q}\to\frac{\partial^{p+q}f(\mathbf{z})}{\partial z_1^p\partial z_2^q}$$

as $k\to+\infty$ on each compact subset in $\mathcal{D}$.

Theorem 1.3 (Principle of analytic continuation [85, p. 39]). *Let the functions $f_1(\mathbf{z})$ and $f_2(\mathbf{z})$ be holomorphic in the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\mathbb{C}^2$, respectively, and let $\mathcal{D}_1\cap\mathcal{D}_2$ be the domain. Let, further, in a real neighborhood of* *the point* $\mathbf{z}^0$ *from* $\mathcal{D}_1 \cap \mathcal{D}_2$ *the functions* $f_1(\mathbf{z})$ *and* $f_2(\mathbf{z})$ *coincide. Then these functions are an analytic continuation of one another, i.e., there is a unique function* $f(\mathbf{z})$ *that is holomorphic in* $\mathcal{D}_1 \cup \mathcal{D}_2$ *and coincides with* $f_1(\mathbf{z})$ *in* $\mathcal{D}_1$ *and with* $f_2(\mathbf{z})$ *in* $\mathcal{D}_2$*.*

Here, the domain is an open connected subset of $\mathbb{C}^2$.

Uniform convergence (see, [19, § 4] and [10, p. 176]). Let $\mathcal{D}$ be a domain in $\mathbb{C}^2$. A sequence $\{f_k(\mathbf{z})\}_{k\geq 1}$ of functions holomorphic in the domain $\mathcal{D}$ is said to converge uniformly on a compact subset $\mathcal{K}$ of $\mathcal{D}$ if:

(i) there exists $N(\mathcal{K})$ such that $f_k(\mathbf{z})$ is holomorphic in the domain $\mathcal{K}$ for all $k \geq N(\mathcal{K})$, and

(ii) given $\varepsilon > 0$ there exists $N_\varepsilon > N(\mathcal{K})$ such that

$$\sup_{\mathbf{z}\in\mathcal{K}} |f_{k+r}(\mathbf{z}) - f_k(\mathbf{z})| < \varepsilon, \; k \geq N_\varepsilon, \; r \geq 0.$$

The sequence $\{f_k(\mathbf{z})\}_{k\geq 1}$ of functions holomorphic in the domain $\mathcal{D}$ is said to be uniformly bounded on a compact subset $\mathcal{K}$ of $\mathcal{D}$ if there exist $N(\mathcal{K})$ and $M(\mathcal{K})$ such that

$$\sup_{\mathbf{z}\in\mathcal{K}} |f_k(\mathbf{z})| < M(\mathcal{K}), \; k \geq N(\mathcal{K}).$$

A branched continued fraction (1.17) is said to converge uniformly on a compact subset $\mathcal{K}$ of $\mathcal{D}$ if its sequence of approximants $\{f_k(\mathbf{z})\}_{k\geq 0}$ converges uniformly on $\mathcal{K}$.

Let $\mathcal{D}$ be some region in $\mathbb{C}^2$. Here, the region refers to a domain (an open connected set) which may include all, part, or none of its boundary. If for each $\mathbf{z} \in \mathcal{D}$ the branched continued fraction (1.17) converges to the finite value $f(\mathbf{z})$, then, for $k \geq 0$,

$$f(\mathbf{z}) - f_k(\mathbf{z})$$

is called the truncation error of the kth approximant (see, for example, [59]). For $k \geq 0$

$$|f(\mathbf{z}) - f_k(\mathbf{z})| \leq M_k(\mathbf{z})$$

is called a priori bound (or truncation error bound), where $M_k(\mathbf{z}) \geq 0$, $k \geq 0$ and, in addition, $M_k(\mathbf{z}) \to 0$ as $k \to +\infty$ for all $\mathbf{z} \in \mathcal{D}$.

In addition, we will use the following lemma [86, Corollary 2] (see also [10, Lemma 4.41]) and the following theorem [49, Theorem 3] (see also [19, Theorem 2.17] and [16, Theorem 24.2]).

Lemma 1.1. *If* $u \geq \tau > 0$ *and* $\eta^2 \leq 4\xi + 4$, *where* $\xi, \eta \in \mathbb{R}$, *then*

$$\inf_{-\infty < v < +\infty} \operatorname{Re}\left(\frac{\xi + i\eta}{u + iv}\right) = -\frac{\sqrt{\xi^2 + \eta^2} - \xi}{2u}.$$

Theorem 1.4 (Convergence continuation theorem)**.** *Let* $\{f_k(\mathbf{z})\}_{k\geq 1}$ *be a sequence of functions, holomorphic in the domain* $\mathcal{D}$, $\mathcal{D} \subset \mathbb{C}^2$, *which is uniformly bounded on every compact subset of* $\mathcal{D}$. *Let the sequence converge at each point of the set* $\mathcal{E}$, $\mathcal{E} \subset \mathcal{D}$, *which is the real neighborhood of the point* $\mathbf{z}^0$ *in* $\mathcal{D}$, *i.e.,*

$$R(\mathbf{z}^0, \tau) = \{\mathbf{z} \in \mathbb{C}^2 : \ |\mathbf{z} - \mathbf{z}^0| < \tau, \ \operatorname{Im}(\mathbf{z}) = \operatorname{Im}(\mathbf{z}^0)\}, \ \tau > 0.$$

Then, $\{f_k(\mathbf{z})\}_{k\geq 1}$ *converges uniformly on every compact subset of* $\mathcal{D}$ *to a function holomorphic in* $\mathcal{D}$.

1.4. Branched continued fraction representations of functions

The natural two-variable extension of the series (1.9) is the double hypergeometric series. In 1931, J. Horn [37] listed all convergent hypergeometric series of the second order: fourteen complete series, including

$$F_1(a, b_1, b_2; c; \mathbf{z}) = \sum_{p,q=0}^{+\infty} \frac{(a)_{p+q}(b_1)_p(b_2)_q}{(c)_{p+q}} \frac{z_1^p z_2^q}{p!q!}, \ |z_1| < 1, \ |z_2| < 1, \tag{1.18}$$

$$F_2(a, b_1, b_2; c_1, c_2; \mathbf{z}) = \sum_{p,q=0}^{+\infty} \frac{(a)_{p+q}(b_1)_p(b_2)_q}{(c_1)_p(c_2)_q} \frac{z_1^p z_2^q}{p!q!}, \ |z_1| + |z_2| < 1, \tag{1.19}$$

$$F_3(a_1, a_2, b_1, b_2; c; \mathbf{z}) =$$

$$= \sum_{p,q=0}^{+\infty} \frac{(a_1)_p(a_2)_q(b_1)_p(b_2)_q}{(c)_{p+q}} \frac{z_1^p z_2^q}{p!q!}, \ |z_1| < 1, \ |z_2| < 1, \tag{1.20}$$

$$F_4(a,b;c_1,c_2;\mathbf{z}) = \sum_{p,q=0}^{+\infty} \frac{(a)_{p+q}(b)_{p+q}}{(c_1)_p(c_2)_q} \frac{z_1^p z_2^q}{p!q!}, \; |z_1|^{1/2} + |z_2|^{1/2} < 1, \tag{1.21}$$

$$H_3(a,b;c;\mathbf{z}) =$$
$$= \sum_{p,q=0}^{+\infty} \frac{(a)_{2p+q}(b)_q}{(c)_{p+q}} \frac{z_1^p z_2^q}{p!q!}, \; |z_1| < r, \; |z_2| < s, \; r + \left(s - \frac{1}{2}\right)^2 = \frac{1}{4}, \tag{1.22}$$

$$H_4(a,b;c_1,c_2;\mathbf{z}) =$$
$$= \sum_{p,q=0}^{+\infty} \frac{(a)_{2p+q}(b)_q}{(c_1)_p(c_2)_q} \frac{z_1^p z_2^q}{p!q!}, \; |z_1| < r, \; |z_2| < s, \; 4r = (s-1)^2, \tag{1.23}$$

and twenty of their confluent cases, including

$$\mathrm{H}_6(a;c;\mathbf{z}) = \sum_{p,q=0}^{+\infty} \frac{(a)_{2p+q}}{(c)_{p+q}} \frac{z_1^p z_2^q}{p!q!}, \; |z_1| < \frac{1}{4}, \tag{1.24}$$

$$\mathrm{H}_7(a;c_1,c_2;\mathbf{z}) = \sum_{p,q=0}^{+\infty} \frac{(a)_{2p+q}}{(c_1)_p(c_2)_q} \frac{z_1^p z_2^q}{p!q!}, \; |z_1| < \frac{1}{4}, \tag{1.25}$$

where $a, a_1, a_2, b, b_1, b_2 \in \mathbb{C}$ and $c, c_1, c_2 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $\mathbf{z} \in \mathbb{C}^2$. Note that (1.18)–(1.21) are also called Appell's hypergeometric series [88]. A list of systems of two partial differential equations, whose solutions are these thirty-four functions, can be found in [38, Section 5.9]. In particular, the solutions of partial differential equations

$$\begin{cases} z_1(1-4z_1)\dfrac{\partial^2 u}{\partial z_1^2} - 4z_1z_2\dfrac{\partial^2 u}{\partial z_1 \partial z_2} - z_2^2\dfrac{\partial^2 u}{\partial z_2^2} + (c_1 - (4a+4)z_1)\dfrac{\partial u}{\partial z_1} \\ \quad -(3a+2)z_2\dfrac{\partial u}{\partial z_2} - a(a+1)u = 0, \\ -2z_1z_2\dfrac{\partial^2 u}{\partial z_1 \partial z_2} + z_2(1-z_2)\dfrac{\partial^2 u}{\partial z_2^2} - 2bz_1\dfrac{\partial u}{\partial z_1} + (c_2 - (a+b)z_2)\dfrac{\partial u}{\partial z_2} - abu = 0 \end{cases}$$

are expressed by means of Horn's hypergeometric function H_4 [37] (see also [38, p. 235]), where u is the unknown function of $\mathbf{z}$. For more on hypergeometric functions from Horn's list, see, for example, [38, Sections 5.9–5.12], [39, Chapter 1], and [87, Chapter 9]. In particular, the recurrence relations for the function (1.23) can be found in [37, 38, 89, 90].

First expansions of the ratios of hypergeometric series

$$\frac{F_1(a;b_1,b_2;c;\mathbf{z})}{F_1(a+1;b_1+\delta_1^j,b_2+\delta_2^j;c+1;\mathbf{z})},\ j=1,2. \tag{1.26}$$

into formal branched continued fractions were considered in [40], where δ_i^j is the Kronecker symbol. However, no explicit formulas for the coefficients of the constructed branched continued fractions were found, nor the criteria of their convergence were established. Later, in [41], the explicit formulas for such expansions was obtained. Other formal expansions for (1.26) can be found in [34]. Of all the constructed branched continued fractions, it was only proven that for each $i=1,2$ the expansion (see, [42])

$$v_0^{(i)}(\mathbf{z})+\sum_{i_1=1}^{2}\cfrac{u_{i(1)}^{(i)}(\mathbf{z})}{v_{i(1)}^{(i)}(\mathbf{z})+\sum_{i_2=1}^{2}\cfrac{u_{i(2)}^{(i)}(\mathbf{z})}{v_{i(2)}^{(i)}(\mathbf{z})+\sum_{i_3=1}^{2}\cfrac{u_{i(3)}^{(i)}(\mathbf{z})}{v_{i(3)}^{(i)}(\mathbf{z})+\ddots}}}$$

converges to a function whose expansion is, and that it provides an analytic extension of this function into the domain

$$\mathcal{D}=\left\{\mathbf{z}\in\mathbb{C}^2:\ \mathrm{Re}(z_1)<\frac{1}{2},\ \mathrm{Re}(z_2)<\frac{1}{2}\right\}$$

under the conditions $a\ge 0$, $b_1\ge 0$, $b_2\ge 0$, $2c\ge a+b_1+b_2+1$, where

$$v_0^{(i)}(\mathbf{z})=1-\frac{a+1}{c}z_i-\sum_{r=1}^{2}\frac{b_r}{c}z_r,$$

$$u_{i(k)}^{(i)}(\mathbf{z})=\frac{(a+k)(b_{i_k}+\varrho_{i(k)})}{(c+k-1)(c+k)}z_{i_k}(1-z_{i_k}),\ i(k)\in\mathcal{I}_k,\ k\ge 1,$$

$$v_{i(k)}^{(i)}(\mathbf{z})=1-\frac{a+k}{c+k}z_{i_k}-\sum_{r=1}^{2}\frac{b_r+\varrho_{i(k),r}}{c+k}z_r,\ i(k)\in\mathcal{I}_k,\ k\ge 1,$$

$$\varrho_{i(1)}=\delta_{i_1}^i,\ i(1)\in\mathcal{I}_1,\ \varrho_{i(k)}=\sum_{r=1}^{k-1}\delta_{i_r}^{i_k}+\delta_{i_k}^i,\ i(k)\in\mathcal{I}_k,\ k\ge 2.$$

Expansions of the ratios of hypergeometric series (1.19) into formal branched continued fractions can be found in [33, 43]. In [44–46] a special case of a branched continued fraction, constructed in [43], was investigated. In particular, the following result was obtained.

Theorem 1.5 ([45, Theorem 7]). *Let a, b_2, and c_2 be real constants such that $0 < u_k \le \tau$, $k \ge 1$, where*

$$u_{2k-1} = \frac{(b_2+k-1)(c_2-a+k-1)}{(c_2+2k-2)(c_2+2k-1)}, \; u_{2k} = \frac{(a+k)(c_2-b_2+k)}{(c_2+2k-1)(c_2+2k)}, \; k \ge 1,$$

τ is a positive number. Then the branched continued fraction

$$1 - z_1 - \cfrac{u_1 z_2}{1 - \cfrac{u_2 z_2}{1 - z_1 - \cfrac{u_3 z_2}{1 - \cfrac{u_4 z_2}{1 - \ddots}}}}$$

converges uniformly on every compact subset of the domain

$$\mathcal{D}_\tau = \left\{ \mathbf{z} \in \mathbb{C}^2 : \; z_1 \notin [1, +\infty), \; z_2 \notin \left[\frac{1}{4\tau}, +\infty\right)\right\}$$

to a function $f(\mathbf{z})$ holomorphic in $\mathcal{D}_\tau$, and, in addition, the $f(\mathbf{z})$ is an analytic continuation of function

$$\frac{F_2(a, b_1, b_2; b_1, c_2; \mathbf{z})}{F_2(a+1, b_1, b_2; b_1, c_2+1; \mathbf{z})}$$

in the domain $\mathcal{D}_\tau$.

The following theorem was proved in [33].

Theorem 1.6. *Let a_1, a_2, b_1, b_2, and c be real positive numbers such that*

$$c > \max\left\{ b_2 + \frac{a_1+b_1+1}{2}, \; b_1 + \frac{a_2+b_2+1}{2} \right\}.$$

Then:

(A) The branched continued fraction

$$v_0(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{u_{i(3)}(\mathbf{z})}{v_{i(3)}(\mathbf{z}) + \ddots}}}$$

converges to a finite value $f(\mathbf{z})$ *for each* $\mathbf{z}\in\mathcal{D}$, *where*

$$\mathcal{D}=\left\{\mathbf{z}\in\mathbb{C}^3:\ \frac{1}{2}-\mathrm{Re}(z_k)\geq\sqrt{|z_k||1-z_k|},\ k=1,2,\right\}$$

$$v_0(\mathbf{z})=1-\frac{a_1+b_1+1}{c}z_1,$$

$$u_{i(1)}(\mathbf{z})=\frac{(a_{i_1}+\delta_{i_1}^1)(b_{i_1}+\delta_{i_1}^1)}{c(c+1)}z_{i_1}(1-\delta_{i_1}^1 z_{i_1}),\ i(1)\in\mathcal{I}_1,$$

$$u_{i(k)}(\mathbf{z})=\frac{(a_{i_k}+p_{i(k)}-\delta_{i_k}^2)(b_{i_k}+p_{i(k)}-\delta_{i_k}^2)}{(c+n-1)(c+n)}z_{i_k}(1-\delta_{i_{k-1}}^{i_k}z_{i_k}),$$

$$i(k)\in\mathcal{I}_k,\ k\geq 2,$$

$$v_{i(k)}(\mathbf{z})=1-\frac{a_{i_k}+b_{i_k}+2p_{i(k)}+(-1)^{\delta_{i_k}^2}}{c+k},\ i(k)\in\mathcal{I}_k,\ k\geq 1,$$

$$p_{i(k)}=\sum_{r=1}^{k}\delta_{i_r}^{i_k},\ i(k)\in\mathcal{I}_k,\ k\geq 1.$$

(B) The convergence is uniform on every compact subset of $\mathrm{Int}(\mathcal{D})$, *and* $f(\mathbf{z})$ *is analytic on* $\mathrm{Int}(\mathcal{D})$, *where* $\mathrm{Int}(\mathcal{D})$ *is the interior of the region* $\mathcal{D}$.

(C) The function $f(\mathbf{z})$ *is an analytic continuation of the function*

$$\frac{F_3(a_1,a_2,b_1,b_2;c;\mathbf{z})}{F_3(a_1+1,a_2,b_1+1,b_2;c+1;\mathbf{z})}$$

in the domain $\mathrm{Int}(\mathcal{D})$.

An expansion of the ratio of the series F_4 into a formal branched continued fraction was constructed in [47]. The expansions of the series ratios (1.22) and (1.24) into branched continued fractions were constructed and investigated in [49, 50] and [51–53], respectively. In [54–56], expansions of the series ratios (1.25) into continued fractions were constructed and investigated.

CHAPTER 2

CONSTRUCTIONS OF BRANCHED CONTINUED FRACTIONS

This chapter considers the Horn hypergeometric series H_4. Contiguous and recurrence relations play a key role in constructing expansions of hypergeometric series ratios into branched continued fractions. In Section 2.1, we establish three- and four-term recurrence relations, and in Section 2.2, we construct formal branched continued fraction expansions.

2.1. Contiguous and recurrence relations

We prove two contiguous recurrence relations for the Horn hypergeometric series (1.23).

Theorem 2.1. *The following assertion holds*

$$H_4(a,b;c_1,c_2;z_1,z_2) - H_4(a,b+1;c_1,c_2;z_1,z_2) = \\ = -\frac{a}{c_2} z_2 H_4(a+1,b+1;c_1,c_2+1;z_1,z_2). \tag{2.1}$$

Proof. By formula (1.23), we obtain

$$H_4(a,b;c_1,c_2;z_1,z_2) - H_4(a,b+1;c_1,c_2;z_1,z_2) = \\ = \sum_{p,q=0}^{\infty} \frac{(a)_{2p+q}(b)_q}{(c_1)_p(c_2)_q} \frac{z_1^p z_2^q}{p!q!} - \sum_{p,q=0}^{\infty} \frac{(a)_{2p+q}(b+1)_q}{(c_1)_p(c_2)_q} \frac{z_1^p z_2^q}{p!q!} = \\ = \sum_{p\geq 0,\ q\geq 1} \frac{(a)_{2p+q}(b+1)_{q-1}}{(c_1)_p(c_2)_q} (b-b-q) \frac{z_1^p z_2^q}{p!q!} = \\ = -\frac{a}{c_2} z_2 \sum_{p\geq 0,\ q\geq 1} \frac{(a+1)_{2p+q-1}(b+1)_{q-1}}{(c_1)_p(c_2+1)_{q-1}} \frac{z_1^p z_2^{q-1}}{p!(q-1)!} = \\ = -\frac{a}{c_2} z_2 \sum_{p\geq 0,\ q\geq 0} \frac{(a+1)_{2p+q}(b+1)_q}{(c_1)_p(c_2+1)_q} \frac{z_1^p z_2^q}{p!q!} = \\ = -\frac{a}{c_2} z_2 H_4(a+1,b+1;c_1,c_2+1;z_1,z_2).$$

Thus, the recurrence relation (2.1) is proved. ■

Theorem 2.2. *The following assertion holds*

$$H_4(a,b;c_1,c_2;z_1,z_2) - H_4(a,b+1;c_1,c_2+1;z_1,z_2) =$$
$$= -\frac{a(c_2-b)}{c_2(c_2+1)} z_2 H_4(a+1,b+1;c_1,c_2+2;z_1,z_2). \tag{2.2}$$

Proof. Using the idea of proving relation (2.1), we obtain

$$H_4(a,b;c_1,c_2;z_1,z_2) - H_4(a,b+1;c_1,c_2+1;z_1,z_2) =$$
$$= \sum_{p\geq 0,\, q\geq 1} \frac{(a)_{2p+q}(b+1)_{q-1}}{(c_1)_p(c_2+1)_{q-1}} \left(\frac{b}{c_2} - \frac{b+q}{c_2+q}\right) \frac{z_1^p z_2^q}{p!q!} =$$
$$= -\sum_{p\geq 0,\, q\geq 1} \frac{(a)_{2p+q}(b+1)_{q-1}}{(c_1)_p(c_2+1)_{q-1}} \frac{q(c_2-b)}{c_2(c_2+q)} \frac{z_1^p z_2^q}{p!q!} =$$
$$= -\frac{a(c_2-b)}{c_2(c_2+1)} z_2 \sum_{p\geq 0,\, q\geq 1} \frac{(a+1)_{2p+q-1}(b+1)_{q-1}}{(c_1)_p(c_2+2)_{q-1}} \frac{z_1^p z_2^{q-1}}{p!(q-1)!} =$$
$$= -\frac{a(c_2-b)}{c_2(c_2+1)} z_2 H_4(a+1,b+1;c_1,c_2+2;z_1,z_2).$$

The theorem is proved. ■

Next, we also prove three four-term recurrence relations for the hypergeometric series (1.23).

Theorem 2.3. *The following assertion holds*

$$H_4(a,b;c_1,c_2;z_1,z_2) - H_4(a+1,b;c_1+1,c_2;z_1,z_2) =$$
$$= -\frac{(2c_1-a)(a+1)}{c_1(c_1+1)} z_1 H_4(a+2,b;c_1+2,c_2;z_1,z_2) -$$
$$-\frac{b}{c_2} z_2 H_4(a+1,b+1;c_1+1,c_2+1;z_1,z_2). \tag{2.3}$$

Proof. In the left part (2.3) using (1.23) and separating the terms at $p=0$, we have

$$H_4(a,b;c_1,c_2;z_1,z_2) - H_4(a+1,b;c_1+1,c_2;z_1,z_2) =$$

$$= \sum_{p,q=0}^{\infty} \frac{(a)_{2p+q}(b)_q}{(c_1)_p(c_2)_q} \frac{z_1^p z_2^q}{p!q!} - \sum_{p,q=0}^{\infty} \frac{(a+1)_{2p+q}(b)_q}{(c_1+1)_p(c_2)_q} \frac{z_1^p z_2^q}{p!q!} =$$

$$= \sum_{q=0}^{\infty} \frac{(a)_q(b)_q}{(c_2)_q} \frac{z_2^q}{q!} - \sum_{q=0}^{\infty} \frac{(a+1)_q(b)_q}{(c_2)_q} \frac{z_2^q}{q!} +$$

$$+ \sum_{p\geq 1,\ q\geq 0} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1+1)_{p-1}(c_2)_q} \left(\frac{a}{c_1} - \frac{a+2p+q}{c_1+p} \right) \frac{z_1^p z_2^q}{p!q!} =$$

$$= \sum_{q=1}^{\infty} \frac{(a+1)_{q-1}(b)_q}{(c_2)_q} (a-a-q) \frac{z_2^q}{q!} +$$

$$+ \sum_{p\geq 1,\ q\geq 0} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1+1)_{p-1}(c_2)_q} \frac{pa-2pc_1-qc_1}{c_1(c_1+p)} \frac{z_1^p z_2^q}{p!q!} =$$

$$= -\frac{b}{c_2} z_2 \sum_{q=1}^{\infty} \frac{(a+1)_{q-1}(b+1)_{q-1}}{(c_2+1)_{q-1}} \frac{z_2^{q-1}}{q!} q + \sum_{p=1}^{\infty} \frac{(a+1)_{2p-1}}{(c_1+1)_{p-1}} \frac{p(a-2c_1)}{c_1(c_1+p)} \frac{z_1^p}{p!} +$$

$$+ \sum_{p\geq 1,\ q\geq 1} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1+1)_{p-1}(c_2)_q} \frac{pa-2pc_1-qc_1}{c_1(c_1+p)} \frac{z_1^p z_2^q}{p!q!} =$$

$$= -\frac{b}{c_2} z_2 \sum_{q=0}^{\infty} \frac{(a+1)_q(b+1)_q}{(c_2+1)_q} \frac{z_2^q}{q!} -$$

$$-\frac{(2c_1-a)(a+1)}{c_1(c_1+1)} z_1 \sum_{p=1}^{\infty} \frac{(a+2)_{2p-2}}{(c_1+2)_{p-1}} \frac{z_1^{p-1}}{(p-1)!} -$$

$$+ \sum_{p\geq 1,\ q\geq 1} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1+1)_{p-1}(c_2)_q} \frac{pa-2pc_1-qc_1}{c_1(c_1+p)} \frac{z_1^p z_2^q}{p!q!} =$$

$$= -\frac{b}{c_2} z_2 \sum_{q=0}^{\infty} \frac{(a+1)_q(b+1)_q}{(c_2+1)_q} \frac{z_2^q}{q!} - \frac{(2c_1-a)(a+1)}{c_1(c_1+1)} z_1 \sum_{p=0}^{\infty} \frac{(a+2)_{2p}}{(c_1+2)_p} \frac{z_1^p}{p!} +$$

$$+ \sum_{p\geq 1,\ q\geq 1} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1+1)_{p-1}(c_2)_q} \frac{pa-2pc_1-qc_1}{c_1(c_1+p)} \frac{z_1^p z_2^q}{p!q!}.$$

Since

$$\sum_{p\geq 1,\ q\geq 1} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1+1)_{p-1}(c_2)_q} \frac{z_1^p z_2^q}{p!q!} \frac{pa-2pc_1-qc_1}{c_1(c_1+p)} =$$

$$= -\frac{(2c_1-a)(a+1)}{c_1(c_1+1)} z_1 \sum_{p\geq 1,\ q\geq 1} \frac{(a+2)_{2(p-1)+q}(b)_q}{(c_1+2)_{p-1}(c_2)_q} \frac{z_1^{p-1}}{(p-1)!} \frac{z_2^q}{q!} -$$

$$-\frac{b}{c_2}z_2\sum_{p\geq 1,\, q\geq 1}\frac{(a+1)_{2p+q-1}(b+1)_{q-1}}{(c_1+1)_p(c_2+1)_{q-1}}\frac{z_1^p z_2^{q-1}}{p!(q-1)!}=$$

$$=-\frac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1\sum_{p\geq 0,\, q\geq 1}\frac{(a+2)_{2p+q}(b)_q}{(c_1+2)_p(c_2)_q}\frac{z_1^p z_2^q}{p!q!}-$$

$$-\frac{b}{c_2}z_2\sum_{p\geq 1,\, q\geq 0}\frac{(a+1)_{2p+q}(b+1)_q}{(c_1+1)_p(c_2+1)_q}\frac{z_1^p z_2^q}{p!q!}$$

and, consequently,

$$-\frac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1\sum_{p=0}^{\infty}\frac{(a+2)_{2p}}{(c_1+2)_p}\frac{z_1^p}{p!}-$$

$$-\frac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1\sum_{p\geq 0,\, q\geq 1}\frac{(a+2)_{2p+q}(b)_q}{(c_1+2)_p(c_2)_q}\frac{z_1^p z_2^q}{p!q!}=$$

$$=-\frac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1\sum_{p\geq 0,\, q\geq 0}\frac{(a+2)_{2p+q}(b)_q}{(c_1+2)_p(c_2)_q}\frac{z_1^p z_2^q}{p!q!},\tag{2.4}$$

and

$$-\frac{b}{c_2}z_2\sum_{q=0}^{\infty}\frac{(a+1)_q(b+1)_q}{(c_2+1)_q}\frac{z_2^q}{q!}-\frac{b}{c_2}z_2\sum_{p\geq 1,\, q\geq 0}\frac{(a+1)_{2p+q}(b+1)_q}{(c_1+1)_p(c_2+1)_q}\frac{z_1^p z_2^q}{p!q!}=$$

$$=-\frac{b}{c_2}z_2\sum_{p\geq 0,\, q\geq 0}\frac{(a+1)_{2p+q}(b+1)_q}{(c_1+1)_p(c_2+1)_q}\frac{z_1^p z_2^q}{p!q!},\tag{2.5}$$

then, adding (2.4) to (2.5), we are convinced of the validity of four-term recurrence relation (2.3). ■

Theorem 2.4. *The following assertion holds*

$$H_4(a,b;c_1,c_2;z_1,z_2)-H_4(a+1,b;c_1,c_2+1;z_1,z_2)=$$

$$=-\frac{2(a+1)}{c_1}z_1H_4(a+2,b;c_1+1,c_2+1;z_1,z_2)-$$

$$-\frac{b(c_2-a)}{c_2(c_2+1)}z_2H_4(a+1,b+1;c_1,c_2+2;z_1,z_2).\tag{2.6}$$

Proof. Using the idea of proving the relation (2.3) in the left part (2.6) with separation of terms at $q = 0$, we get

$$H_4(a,b;c_1,c_2;z_1,z_2) - H_4(a+1,b;c_1,c_2+1;z_1,z_2) =$$

$$= \sum_{p=0}^{\infty} \frac{(a)_{2p}}{(c_1)_p} \frac{z_1^p}{p!} - \sum_{p=0}^{\infty} \frac{(a+1)_{2p}}{(c_1)_p} \frac{z_1^p}{p!} +$$

$$+ \sum_{p\geq 0,\, q\geq 1} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1)_p(c_2+1)_{q-1}} \left(\frac{a}{c_2} - \frac{a+2p+q}{c_2+q} \right) \frac{z_1^p z_2^q}{p!q!} =$$

$$= -\frac{2(a+1)}{c_1} z_1 \sum_{p=1}^{\infty} \frac{(a+2)_{2(p-1)}}{(c_1+1)_{p-1}} \frac{z_1^{p-1}}{(p-1)!} -$$

$$- \sum_{p\geq 0,\, q\geq 1} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1)_p(c_2+1)_{q-1}} \frac{2pc_2 + qc_2 - qa}{c_2(c_2+q)} \frac{z_1^p z_2^q}{p!q!} =$$

$$= -\frac{2(a+1)}{c_1} z_1 \sum_{p=0}^{\infty} \frac{(a+2)_{2p}}{(c_1+1)_p} \frac{z_1^p}{p!} -$$

$$- \sum_{p\geq 0,\, q\geq 1} \frac{(a+1)_{2p+q-1}(b)_q}{(c_1)_p(c_2+1)_{q-1}} \frac{2p}{c_2+q} \frac{z_1^p z_2^q}{p!q!} -$$

$$- \frac{b(c_2-a)}{c_2(c_2+1)} z_2 \sum_{p\geq 0,\, q\geq 1} \frac{(a+1)_{2p+q-1}(b+1)_{q-1}}{(c_1)_p(c_2+2)_{q-1}} \frac{z_1^p z_2^{q-1}}{p!(q-1)!} =$$

$$= -\frac{2(a+1)}{c_1} z_1 \sum_{p=0}^{\infty} \frac{(a+2)_{2p}}{(c_1+1)_p} \frac{z_1^p}{p!} -$$

$$- \frac{2(a+1)}{c_1} z_1 \sum_{p\geq 1,\, q\geq 1} \frac{(a+2)_{2p+q-2}(b)_q}{(c_1)_{p-1}(c_2+1)_q} \frac{z_1^{p-1} z_2^q}{(p-1)!q!} -$$

$$- \frac{b(c_2-a)}{c_2(c_2+1)} z_2 \sum_{p\geq 0,\, q\geq 0} \frac{(a+1)_{2p+q}(b+1)_q}{(c_1)_p(c_2+2)_q} \frac{z_1^p z_2^q}{p!q!} =$$

$$= -\frac{2(a+1)}{c_1} z_1 \sum_{p=0}^{\infty} \frac{(a+2)_{2p}}{(c_1+1)_p} \frac{z_1^p}{p!} -$$

$$- \frac{2(a+1)}{c_1} z_1 \sum_{p\geq 0,\, q\geq 1} \frac{(a+2)_{2p+q}(b)_q}{(c_1+1)_p(c_2+1)_q} \frac{z_1^p z_2^q}{p!q!} -$$

$$- \frac{b(c_2-a)}{c_2(c_2+1)} z_2 \sum_{p\geq 0,\, q\geq 0} \frac{(a+1)_{2p+q}(b+1)_q}{(c_1)_p(c_2+2)_q} \frac{z_1^p z_2^q}{p!q!} =$$

$$= -\frac{2(a+1)}{c_1} z_1 \sum_{p\geq 0,\, q\geq 0} \frac{(a+2)_{2p+q}(b)_q}{(c_1+1)_p(c_2+1)_q} \frac{z_1^p z_2^q}{p!q!} -$$

$$-\frac{b(c_2-a)}{c_2(c_2+1)}z_2\sum_{p\geq 0,\, q\geq 0}\frac{(a+1)_{2p+q}(b+1)_q}{(c)_p(c_2+2)_q}\frac{z_1^p z_2^q}{p!q!}.$$

Finally, in view of formula (1.23), we directly obtain the four-term recurrence relation (2.6). ■

We prove the following theorem.

Theorem 2.5. *The following assertion holds*

$$\begin{aligned}
&H_4(a,b;c_1,c_2;z_1,z_2)-H_4(a+1,b;c_1,c_2;z_1,z_2)=\\
&=-\frac{2(a+1)}{c_1}z_1H_4(a+2,b;c_1+1,c_2;z_1,z_2)-\\
&-\frac{b}{c_2}z_2H_4(a+1,b+1;c_1,c_2+1;z_1,z_2).
\end{aligned}\tag{2.7}$$

Proof. We have

$$\begin{aligned}
&H_4(a,b;c_1,c_2;z_1,z_2)-H_4(a+1,b;c_1,c_2;z_1,z_2)=\\
&=\sum_{p,q=0}^{\infty}\frac{(a)_{2p+q}(b)_q}{(c_1)_p(c_2)_q}\frac{z_1^p z_2^q}{p!q!}-\sum_{p,q=0}^{\infty}\frac{(a+1)_{2p+q}(b)_q}{(c_1)_p(c_2)_q}\frac{z_1^p z_2^q}{p!q!}=\\
&=\sum_{q=0}^{\infty}\frac{(a)_q(b)_q}{(c_2)_q}\frac{z_2^q}{q!}-\sum_{q=0}^{\infty}\frac{(a+1)_q(b)_q}{(c_2)_q}\frac{z_2^q}{q!}+\\
&+\sum_{p\geq 1,\, q\geq 0}\frac{(a+1)_{2p+q-1}(b)_q}{(c_1)_p(c_2)_q}(a-a-2p-q)\frac{z_1^p z_2^q}{p!q!}=\\
&=-\sum_{q=1}^{\infty}\frac{(a+1)_{q-1}(b)_q}{(c_2)_q}q\frac{z_2^q}{q!}-\sum_{p\geq 1,\, q\geq 0}\frac{(a+1)_{2p+q-1}(b)_q}{(c_1)_p(c_2)_q}(2p+q)\frac{z_1^p z_2^q}{p!q!}=\\
&=-\frac{b}{c_2}z_2\sum_{q=1}^{\infty}\frac{(a+1)_{q-1}(b+1)_{q-1}}{(c_2+1)_{q-1}}\frac{z_2^{q-1}}{(q-1)!}\\
&-\sum_{p\geq 1,\, q\geq 0}\frac{(a+1)_{2p+q-1}(b)_q}{(c_1+1)_{p-1}(c_2)_q}\frac{2}{c_1}\frac{z_1^p}{(p-1)!}\frac{z_2^q}{q!}-\\
&-\sum_{p\geq 1,\, q\geq 1}\frac{(a+1)_{2p+q-1}(b+1)_{q-1}}{(c_1)_p(c_2+1)_{q-1}}\frac{b}{c_2}\frac{z_1^p}{p!}\frac{z_2^q}{(q-1)!}=\\
&=-\frac{b}{c_2}z_2\sum_{q=0}^{\infty}\frac{(a+1)_q(b+1)_q}{(c_2+1)_q}\frac{z_2^q}{q!}-
\end{aligned}$$

$$-\frac{2(a+1)}{c_1}z_1\sum_{p\geq 1,\, q\geq 0}\frac{(a+2)_{2p+q-2}(b)_q}{(c_1+1)_{p-1}(c_2)_q}\frac{z_1^{p-1}z_2^q}{(p-1)!q!}-$$

$$-\frac{b}{c_2}z_2\sum_{p\geq 1,\, q\geq 1}\frac{(a+1)_{2p+q-1}(b+1)_{q-1}}{(c_1)_p(c_2+1)_{q-1}}\frac{z_1^p z_2^{q-1}}{p!(q-1)!}=$$

$$=-\frac{2(a+1)}{c_1}z_1\sum_{p\geq 0,\, q\geq 0}\frac{(a+2)_{2p+q}(b)_q}{(c_1+1)_p(c_2)_q}\frac{z_1^p z_2^q}{p!q!}-$$

$$-\frac{b}{c_2}z_2\sum_{p\geq 0,\, q\geq 0}\frac{(a+1)_{2p+q}(b+1)_q}{(c_1)_p(c_2+1)_q}\frac{z_1^p z_2^q}{p!q!}.$$

Finally, using the formula (1.23), we obtain the relation (2.7). ■

2.2. Branched continued fraction expansions

We consider the following ratios

$$\frac{H_4(a,b;c_1,c_2;\mathbf{z})}{H_4(a+1,b;c_1+1,c_2;\mathbf{z})},\ \frac{H_4(a,b;c_1,c_2;\mathbf{z})}{H_4(a+1,b;c_1,c_2+1;\mathbf{z})},$$

and

$$\frac{H_4(a,b;c_1,c_2;\mathbf{z})}{H_4(a,b+1;c_1,c_2+1;\mathbf{z})}.$$

Let $(ij)_0=(i_0,j_0)$ be a double index and

$$\mathcal{I}=\{(1,1);(1,2);(2,2)\}$$

be a set of double indices. Then, for each $(ij)_0\in\mathcal{I}$ we set

$$R_{(ij)_0}(a,b;c_1,c_2;\mathbf{z})=\frac{H_4(a,b;c_1,c_2;\mathbf{z})}{H_4(a+\delta_{i_0}^1,b+\delta_{i_0}^2;c_1+\delta_{j_0}^1,c_2+\delta_{j_0}^2;\mathbf{z})}, \tag{2.8}$$

where δ_i^j is the Kronecker symbol. Now, let $(ij)_k=(i_1,j_1,i_2,j_2,\ldots,i_k,j_k)$ be a multiindex. Then, for each pair $(i_0,j_0)\in\mathcal{I}$ we introduce the following sets of multiindices

$$\mathcal{I}_k^{(ij)_0}=\{(ij)_k:\ 1\leq i_k\leq 2-\delta_{i_{k-1}}^2,\ j_k=i_k+\delta_{i_{k-1}}^2\},\ k\geq 1.$$

Example 2.1. *If* $(ij)_0 = (1,1)$, *then*

$$\mathcal{I}_1^{1,1} = \{(1,1);(2,2)\},\ \mathcal{I}_2^{1,1} = \{(1,1,1,1);(1,1,2,2);(2,2,1,2)\},$$
$$\mathcal{I}_3^{1,1} = \{(1,1,1,1,1,1);(1,1,1,1,2,2);(1,1,2,2,1,2);$$
$$(2,2,1,2,1,1);(2,2,1,2,2,2)\},\ \textit{etc.}$$

Next, for each $(ij)_k \in \mathcal{I}_k^{(ij)_0}$, $k \geq 1$, we set

$$a_{(ij)_k}^{(ij)_0} = a + \sum_{r=0}^{k-1} \delta_{i_r}^1,\ b_{(ij)_k}^{(ij)_0} = b + \sum_{r=0}^{k-1} \delta_{i_r}^2, \tag{2.9}$$

and

$$\mu_{(ij)_k}^{(ij)_0} = c_1 + \sum_{r=0}^{k-1} \delta_{j_r}^1,\ \nu_{(ij)_k}^{(ij)_0} = c_2 + \sum_{r=0}^{k-1} \delta_{j_r}^2. \tag{2.10}$$

We prove the following theorem.

Theorem 2.6. *For each* $(ij)_0 \in \mathcal{I}$ *the ratio* (2.8) *has a formal branched continued fraction*

$$1 - \sum_{\substack{i_1=1\\ j_1=i_1+\delta_{i_0}^2}}^{2-\delta_{i_0}^2} \frac{u_{(ij)_1}^{(ij)_0} z_{j_1}}{1} - \sum_{\substack{i_2=1\\ j_2=i_2+\delta_{i_1}^2}}^{2-\delta_{i_1}^2} \frac{u_{(ij)_2}^{(ij)_0} z_{j_2}}{1} - \sum_{\substack{i_3=1\\ j_3=i_3+\delta_{i_2}^2}}^{2-\delta_{i_2}^2} \frac{u_{(ij)_3}^{(ij)_0} z_{j_3}}{1} - \ldots, \tag{2.11}$$

where for $(ij)_k \in \mathcal{I}_k^{(ij)_0}$, $k \geq 1$, $(ij)_0 \in \mathcal{I}$,

$$u_{(ij)_k}^{(ij)_0} = \frac{\left(2c_1 - a + \sum_{r=0}^{k-2}(2\delta_{j_r}^1 - \delta_{i_r}^1)\right)\left(a + 1 + \sum_{r=0}^{k-2}\delta_{i_r}^1\right)}{\left(c_1 + \sum_{r=0}^{k-2}\delta_{j_r}^1\right)\left(c_1 + 1 + \sum_{r=0}^{k-2}\delta_{j_r}^1\right)} \tag{2.12}$$

if $i_{k-1} = j_{k-1} = i_k = j_k = 1$,

$$u_{(ij)_k}^{(ij)_0} = \frac{b + \sum_{r=0}^{k-2}\delta_{i_r}^2}{c_2 + \sum_{r=0}^{k-2}\delta_{j_r}^2}, \tag{2.13}$$

if $i_{k-1} = j_{k-1} = 1$, $i_k = j_k = 2$,

$$u^{(ij)_0}_{(ij)_k} = \frac{2\left(a+1+\sum_{r=0}^{k-2}\delta^1_{i_r}\right)}{c_1+\sum_{r=0}^{k-2}\delta^1_{j_r}} \tag{2.14}$$

if $j_{k-1} = 2$, $i_{k-1} = i_k = j_k = 1$,

$$u^{(ij)_0}_{(ij)_k} = \frac{\left(b+\sum_{r=0}^{k-2}\delta^2_{i_r}\right)\left(c_2-a+\sum_{r=0}^{k-2}(\delta^2_{j_r}-\delta^1_{i_r})\right)}{\left(c_2+\sum_{r=0}^{k-2}\delta^2_{j_r}\right)\left(c_2+1+\sum_{r=0}^{k-2}\delta^2_{j_r}\right)} \tag{2.15}$$

if $i_{k-1} = 1$, $j_{k-1} = i_k = j_k = 2$,

$$u^{(ij)_0}_{(ij)_k} = \frac{\left(a+\sum_{r=0}^{k-2}\delta^1_{i_r}\right)\left(c_2-b+\sum_{r=0}^{k-2}(\delta^2_{j_r}-\delta^1_{i_r})\right)}{\left(c_2+\sum_{r=0}^{k-2}\delta^2_{j_r}\right)\left(c_2+1+\sum_{r=0}^{k-2}\delta^2_{j_r}\right)} \tag{2.16}$$

if $j_{k-1} = i_{k-1} = j_k = 2$, $i_k = 1$.

Proof. Dividing the relation (2.3) by $H_4(a+1,b;c_1+1,c_2;\mathbf{z})$, (2.6) by $H_4(a+1,b;c_1,c_2+1;\mathbf{z})$, and (2.2) by $H_4(a,b+1;c_1,c_2+1;\mathbf{z})$, we obtain

$$R_{1,1}(a,b;c_1,c_2;\mathbf{z}) =$$
$$= 1 - \frac{\frac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1}{R_{1,1}(a+1,b;c_1+1,c_2;\mathbf{z})} - \frac{\frac{b}{c_2}z_2}{R_{2,2}(a+1,b;c_1+1,c_2;\mathbf{z})}, \tag{2.17}$$

$$R_{1,2}(a,b;c_1,c_2;\mathbf{z}) =$$
$$= 1 - \frac{\frac{2(a+1)}{c_1}z_1}{R_{1,1}(a+1,b;c_1,c_2+1;\mathbf{z})} - \frac{\frac{b(c_2-a)}{c_2(c_2+1)}z_2}{R_{2,2}(a+1,b;c_1,c_2+1;\mathbf{z})}, \tag{2.18}$$

and

$$R_{2,2}(a,b;c_1,c_2;\mathbf{z}) = 1 - \frac{\frac{a(c_2-b)}{c_2(c_2+1)}z_2}{R_{1,2}(a,b+1;c_1,c_2+1;\mathbf{z})}, \tag{2.19}$$

respectively.

Hence, for any $(ij)_0 \in \mathcal{I}$ it follows

$$R_{(ij)_0}(a,b;c_1,c_2;\mathbf{z}) =$$
$$= 1 - \sum_{\substack{i_1=1\\ j_1=i_1+\delta_{i_0}^2}}^{2-\delta_{i_0}^2} \frac{u_{(ij)_1}^{(ij)_0} z_{j_1}}{R_{i_1,j_1}(a+\delta_{i_0}^1, b+\delta_{i_0}^2; c_1+\delta_{j_0}^1, c_2+\delta_{j_0}^2;\mathbf{z})}, \tag{2.20}$$

where $u_{(ij)_1}^{(ij)_0}$, $(ij)_1 \in \mathcal{I}_1^{(ij)_0}$, $(ij)_0 \in \mathcal{I}$, are defined by (2.12)–(2.16). Furthermore, this is the first step to constructing branched continued fraction expansions.

By analogy, it is clear that for

$$(ij)_{k-1} \in \mathcal{I}_{k-1}^{(ij)_0},\ k \geq 2,\ (ij)_0 \in \mathcal{I},$$

the following relation holds

$$R_{i_{k-1},j_{k-1}}(a_{(ij)_{k-1}}^{(ij)_0}, b_{(ij)_{k-1}}^{(ij)_0}; \mu_{(ij)_{k-1}}^{(ij)_0}, \nu_{(ij)_{k-1}}^{(ij)_0};\mathbf{z}) =$$
$$= 1 - \sum_{\substack{i_k=1\\ j_k=i_k+\delta_{i_{k-1}}^2}}^{2-\delta_{i_{k-1}}^2} \frac{u_{(ij)_k}^{(ij)_0} z_{j_k}}{R_{i_k,j_k}(a_{(ij)_k}^{(ij)_0}, b_{(ij)_k}^{(ij)_0}; \mu_{(ij)_k}^{(ij)_0}, \nu_{(ij)_k}^{(ij)_0};\mathbf{z})}, \tag{2.21}$$

where $u_{(ij)_k}^{(ij)_0}$, $(ij)_k \in \mathcal{I}_k^{(ij)_0}$, $k \geq 2$, $(ij)_0 \in \mathcal{I}$, are defined by (2.12)–(2.16), $a_{(ij)_k}^{(ij)_0}$, $b_{(ij)_k}^{(ij)_0}$ and $\mu_{(ij)_k}^{(ij)_0}$, $\nu_{(ij)_k}^{(ij)_0}$, $(ij)_k \in \mathcal{I}_k^{(ij)_0}$, $k \geq 1$, $(ij)_0 \in \mathcal{I}$, are defined by (2.9) and (2.10), respectively.

Substituting relation (2.21) at $k = 2$ in formula (2.20) on the second step for any $(ij)_0 \in \mathcal{I}$ we obtain

$$R_{(ij)_0}(a,b;c_1,c_2;\mathbf{z}) =$$
$$= 1 - \sum_{\substack{i_1=1\\ j_1=i_1+\delta_{i_0}^2}}^{2-\delta_{i_0}^2} \frac{u_{(ij)_1}^{(ij)_0} z_{j_1}}{1 - \sum\limits_{\substack{i_2=1\\ j_2=i_2+\delta_{i_1}^2}}^{2-\delta_{i_1}^2} \dfrac{u_{(ij)_2}^{(ij)_0} z_{j_2}}{R_{i_2,j_2}(a_{(ij)_2}^{(ij)_0}, b_{(ij)_2}^{(ij)_0}; \mu_{(ij)_2}^{(ij)_0}, \nu_{(ij)_2}^{(ij)_0};\mathbf{z})}}.$$

Next, applying recurrence relation (2.21) after n steps, we have

$$R_{(ij)_0}(a,b;c_1,c_2;\mathbf{z}) =$$

$$= 1 - \sum_{\substack{i_1=1\\ j_1=i_1+\delta_{i_0}^2}}^{2-\delta_{i_0}^2} \frac{u_{(ij)_1}^{(ij)_0} z_{j_1}}{1} - \sum_{\substack{i_2=1\\ j_2=i_2+\delta_{i_1}^2}}^{2-\delta_{i_1}^2} \frac{u_{(ij)_2}^{(ij)_0} z_{j_2}}{1} - \cdots - \sum_{\substack{i_{n-1}=1\\ j_{n-1}=i_{n-1}+\delta_{i_{n-2}}^2}}^{2-\delta_{i_{n-2}}^2} \frac{u_{(ij)_{n-1}}^{(ij)_0} z_{j_{n-1}}}{1} -$$

$$- \sum_{\substack{i_n=1\\ j_n=i_n+\delta_{i_{n-1}}^2}}^{2-\delta_{i_{n-1}}^2} \frac{u_{(ij)_n}^{(ij)_0} z_{j_n}}{R_{i_n,j_n}(a_{(ij)_n}^{(ij)_0}, b_{(ij)_n}^{(ij)_0}; \mu_{(ij)_n}^{(ij)_0}, \nu_{(ij)_n}^{(ij)_0}; \mathbf{z})}, \tag{2.22}$$

where $u_{(ij)_k}^{(ij)_0}$, $(ij)_k \in \mathcal{I}_k^{(ij)_0}$, $1 \le k \le n$, $(ij)_0 \in \mathcal{I}$, are defined by (2.12)–(2.16), $a_{(ij)_n}^{(ij)_0}$, $b_{(ij)_n}^{(ij)_0}$ and $\mu_{(ij)_n}^{(ij)_0}$, $\nu_{(ij)_n}^{(ij)_0}$, $(ij)_n \in \mathcal{I}_n^{(ij)_0}$, $(ij)_0 \in \mathcal{I}$, are defined by (2.9) and (2.10), respectively.

Finally, by the recurrence relation (2.21), from (2.22) we obtain the formal branched continued fraction (2.11) for the ratio (2.8) for each $(ij)_0 \in \mathcal{I}$. ■

Note that there are three different branched continued fractions in (2.11).

Example 2.2. *The ratio $R_{1,1}(a,b;c_1,c_2;\mathbf{z})$ has a formal branched continued fraction*

$$1 - \cfrac{c_{1,1}^{1,1} z_1}{1 - \cfrac{c_{1,1,1,1}^{1,1} z_1}{1 - \cfrac{c_{1,1,1,1,1,1}^{1,1} z_1}{1 - \ddots} - \cfrac{c_{1,1,1,1,2,2}^{1,1} z_2}{1 - \ddots}} - \cfrac{c_{1,1,2,2}^{1,1} z_2}{1 - \cfrac{c_{1,1,2,2,1,2}^{1,1} z_2}{1 - \ddots}}} - \cfrac{c_{2,2}^{1,1} z_2}{1 - \cfrac{c_{2,2,1,2}^{1,1} z_2}{1 - \cfrac{c_{2,2,1,2,1,1}^{1,1} z_1}{1 - \ddots} - \cfrac{c_{2,2,1,2,2,2}^{1,1} z_2}{1 - \ddots}}}.$$

Next, we will consider the following special cases

$$\frac{H_4(a,c_2;c_1,c_2;\mathbf{z})}{H_4(a+1,c_2;c_1+1,c_2;\mathbf{z})}, \tag{2.23}$$

$$\frac{H_4(a,c_2+1;c_1,c_2;\mathbf{z})}{H_4(a+1,c_2+1;c_1,c_2+1;\mathbf{z})}, \tag{2.24}$$

and

$$\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})}. \tag{2.25}$$

The following theorem holds.

Theorem 2.7. *The ratio* (2.23) *has a formal branched continued fraction*

$$1 - z_2 - \cfrac{u_1 z_1}{1 - z_2 - \cfrac{u_2 z_1}{1 - z_2 - \cfrac{u_3 z_1}{1 - \ddots}}}, \tag{2.26}$$

where

$$u_k = \frac{(2c_1 - a + k - 1)(a + k)}{(c_1 + k - 1)(c_1 + k)}, \ k \geq 1. \tag{2.27}$$

Proof. Let $b = c_2$ and $(ij)_0 = (1,1)$ in Theorem 2.6. Then, in the first step, from (2.17) we get

$$R_{1,1}(a, c_2; c_1, c_2; \mathbf{z}) =$$

$$= 1 - \frac{\frac{(2c_1 - a)(a+1)}{c_1(c_1+1)} z_1}{R_{1,1}(a+1, c_2; c_1+1, c_2; \mathbf{z})} - \frac{z_2}{R_{2,2}(a+1, c_2; c_1+1, c_2; \mathbf{z})}. \tag{2.28}$$

Replacing a by $a+1$ and c_1 by c_1+1 in (2.17), in the second step we have

$$R_{1,1}(a+1, c_2; c_1+1, c_2; \mathbf{z}) =$$

$$= 1 - \frac{\frac{(2(c_1+1) - (a+1))((a+1)+1)}{(c_1+1)((c_1+1)+1)} z_1}{R_{1,1}(a+2, c_2; c_1+2, c_2; \mathbf{z})} -$$

$$- \frac{z_2}{R_{2,2}(a+2, c_2; c_1+2, c_2; \mathbf{z})} = 1 - \frac{\frac{(2c_1 - a + 1)(a+2)}{(c_1+1)(c_1+2)} z_1}{R_{1,1}(a+2, c_2; c_1+2, c_2; \mathbf{z})} -$$

$$- \frac{z_2}{R_{2,2}(a+2, c_2; c_1+2, c_2; \mathbf{z})}.$$

Again replacing a by $a+1$ and c_1 by c_1+1, but now in (2.19), we get

$$R_{2,2}(a+1, c_2; c_1+1, c_2; \mathbf{z}) = 1 - \frac{\frac{(a+1)(c_2 - c_2)}{c_2(c_2+1)} z_2}{R_{1,2}(a+1, c_2+1; c_1+1, c_2+1; \mathbf{z})} = 1.$$

Next, substituting the obtained results into (2.28), we obtain that

$$R_{1,1}(a,c_2;c_1,c_2;\mathbf{z}) = 1 - z_2 - \cfrac{\cfrac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1}{1-\cfrac{\cfrac{(2c_1-a+1)(a+2)}{(c_1+1)(c_1+2)}z_1}{R_{1,1}(a+2,c_2;c_1+2,c_2;\mathbf{z})}-\cfrac{z_2}{R_{2,2}(a+2,c_2;c_1+2,c_2;\mathbf{z})}}. \tag{2.29}$$

By analogy, for $k \geq 1$ the following relations hold

$$R_{1,1}(a+k-1,c_2;c_1+k-1,c_2;\mathbf{z}) = \\ = 1 - \frac{z_2}{R_{2,2}(a+k,c_2;c_1+k,c_2;\mathbf{z})} - \cfrac{\cfrac{(2c_1-a+k-1)(a+k)}{(c_1+k-1)(c_1+k)}z_1}{R_{1,1}(a+k,c_2;c_1+k,c_2;\mathbf{z})}. \tag{2.30}$$

and

$$R_{2,2}(a+k,c_2;c_1+k,c_2;\mathbf{z}) = 1. \tag{2.31}$$

Applying the relations (2.30) and (2.31), after n step we obtain

$$R_{1,1}(a,c_2;c_1,c_2;\mathbf{z}) = \\ = 1 - z_2 - \cfrac{\cfrac{(2c_1-a)(a+1)}{c_1(c_1+1)}z_1}{1-z_2-\cfrac{\cfrac{(2c_1-a+1)(a+2)}{(c_1+1)(c_1+2)}z_1}{1-\ddots -z_2-\cfrac{\cfrac{(2c_1-a+n-1)(a+n)}{(c_1+n-1)(c_1+n)}z_1}{R_{1,1}(a+n,c_2;c_1+n,c_2;\mathbf{z})}}}. \tag{2.32}$$

Finally, by (2.30) and (2.31), from (2.32) we obtain the branched continued fraction (2.26) with coefficients defined as (2.27) for the ratio (2.23). ■

If we set $b = c_2+1$ and choose $(ij)_0 = (1,2)$ in Theorem 2.6, the following result can be proved in much the same way as Theorem 2.7.

Theorem 2.8. *The ratio* (2.24) *has a formal branched continued fraction*

$$1 - \frac{c_2-a}{c_2}z_2 - \cfrac{v_1 z_1}{1-z_2-\cfrac{v_2 z_1}{1-z_2-\cfrac{v_3 z_1}{1-\ddots}}}, \tag{2.33}$$

where

$$v_1 = \frac{2(a+1)}{c_1},\ v_k = \frac{(2c_1 - a + k - 3)(a+k)}{(c_1 + k - 2)(c_1 + k - 1)},\ k \geq 2. \tag{2.34}$$

Setting $a = 0$ and replacing c_1 by $c_1 - 1$ in Theorem 2.7 or setting $a = 0$ and replacing c_2 by $c_2 - 1$ in Theorem 2.8, we have the following result.

Corollary 2.1. *The function*

$$H_4(1, c_2; c_1, c_2; \mathbf{z}) \tag{2.35}$$

has a formal branched continued fraction

$$\cfrac{1}{1 - z_2 - \cfrac{w_1 z_1}{1 - z_2 - \cfrac{w_2 z_1}{1 - \ddots}}}, \tag{2.36}$$

where

$$w_k = \frac{k(2c_1 + k - 3)}{(c_1 + k - 2)(c_1 + k - 1)},\ k \geq 1. \tag{2.37}$$

Finally, we prove the following theorem.

Theorem 2.9. *The ratio* (2.25) *has a formal branched continued fraction*

$$1 + \cfrac{d_0 z_2}{1 - d_1 z_2 - \cfrac{h_1 z_1}{1 - d_2 z_2 - \cfrac{h_2 z_1}{1 - \ddots}}}, \tag{2.38}$$

where

$$d_0 = \frac{a}{c_2(c_2+1)},\ d_1 = 1 - \frac{a}{c_2 + 1},\ h_1 = \frac{2(a+1)}{c_1}, \tag{2.39}$$

$$d_k = 1,\ h_k = \frac{(2c_1 - a + k - 3)(a+k)}{(c_1 + k - 2)(c_1 + k - 1)},\ k \geq 2. \tag{2.40}$$

Proof. Let $b = c_2 + 1$ and $(ij)_0 = (2, 2)$ in Theorem 2.6. Then, in the first step, from (2.19) we obtain

$$R_{2,2}(a, c_2 + 1; c_1, c_2; \mathbf{z}) = 1 + \frac{\dfrac{a}{c_2(c_2+1)} z_2}{R_{1,2}(a, c_2 + 2; c_1, c_2 + 1; \mathbf{z})}. \tag{2.41}$$

In the second step, replacing c_2 by c_2+1 and b by c_2+2 in the relation (2.18), we have

$$R_{1,2}(a, c_2+2; c_1, c_2+1; \mathbf{z}) =$$
$$= 1 - \cfrac{\frac{2(a+1)}{c_1} z_1}{R_{1,1}(a+1, c_2+2; c_1, c_2+2; \mathbf{z})} - \cfrac{\frac{c_2+1-a}{c_2+1} z_2}{R_{2,2}(a+1, c_2+2; c_1, c_2+2; \mathbf{z})}.$$

Substituting the obtained results into (2.41), we obtain that

$$R_{2,2}(a, c_2+1; c_1, c_2; \mathbf{z}) =$$
$$= 1 + \cfrac{\frac{a}{c_2(c_2+1)} z_2}{1 + \cfrac{\left(\frac{a}{c_2+1} - 1\right) z_2}{R_{2,2}(a+1, c_2+2; c_1, c_2+2; \mathbf{z})} - \cfrac{\frac{2(a+1)}{c_1} z_1}{R_{1,1}(a+1, c_2+2; c_1, c_2+2; \mathbf{z})}}.$$

In the third step, replacing a by $a+1$, b by c_2+2, c_2 by c_2+2 in the relation (2.17), we have

$$R_{1,1}(a+1, c_2+2; c_1, c_2+2; \mathbf{z}) =$$
$$= 1 - \cfrac{\frac{(2c_1-a-1)(a+2)}{c_1(c_1+1)} z_1}{R_{1,1}(a+2, c_2+2; c_1+1, c_2+2; \mathbf{z})} -$$
$$- \frac{z_2}{R_{2,2}(a+2, c_2+2; c_1+1, c_2+2; \mathbf{z})}.$$

Again replacing a by $a+1$, b by c_2+2 and c_2 by c_2+2, but now in (2.19), we get

$$R_{2,2}(a+1, c_2+2; c_1, c_2+2; \mathbf{z}) =$$
$$= 1 - \cfrac{\frac{(a+1)(c_2+2-(c_2+2))}{(c_2+2)(c_2+3)} z_2}{R_{1,2}(a+1, c_2+3; c_1, c_2+3; \mathbf{z})} = 1.$$

Then,

$$R_{2,2}(a, c_2+1; c_1, c_2; \mathbf{z}) = 1 + \cfrac{\cfrac{a}{c_2(c_2+1)}z_2}{1} + $$

$$+\left(\left(\frac{a}{c_2+1}-1\right)z_2 - \cfrac{\cfrac{2(a+1)}{c_1}z_1}{1} - \left(\frac{z_2}{R_{2,2}(a+2, c_2+2; c_1+1, c_2+2; \mathbf{z})} + \right.\right.$$

$$\left.\left. + \cfrac{\cfrac{(2c_1-a-1)(a+2)}{c_1(c_1+1)}z_1}{R_{1,1}(a+2, c_2+2; c_1+1, c_2+2; \mathbf{z})}\right)\right).$$

By analogy, for $k \geq 1$ the following relations hold

$$R_{1,1}(a+k, c_2+2; c_1+k-1, c_2+2; \mathbf{z})$$
$$= 1 - z_2 - \cfrac{\cfrac{(2c_1-a+k-2)(a+k+1)}{(c_1+k-1)(c_1+k)}z_1}{R_{1,1}(a+k+1, c_2+2; c_1+k, c_2+2; \mathbf{z})} \tag{2.42}$$

and

$$R_{2,2}(a+k, c_2+2; c_1+k-1, c_2+2; \mathbf{z}) = 1. \tag{2.43}$$

After step n, using (2.42) and (2.43), we get

$$R_{2,2}(a, c_2+1; c_1, c_2; \mathbf{z}) = 1 + \cfrac{\cfrac{a}{c_2(c_2+1)}z_2}{1} + $$

$$+\left(\left(\frac{a}{c_2+1}-1\right)z_2 - \cfrac{\cfrac{2(a+1)}{c_1}z_1}{1} - \left(z_2 + \cfrac{\cfrac{(2c_1-a-1)(a+2)}{c_1(c_1+1)}z_1}{1} - \cdots - \right.\right.$$

$$\left.\left. - \left(z_2 + \cfrac{\cfrac{(2c_1-a+n-4)(a+n-1)}{(c_1+n-3)(c_1+n-2)}z_1}{R_{1,1}(a+n-1, c_2+2; c_1+n-2, c_2+2; \mathbf{z})}\right)\cdots\right)\right). \tag{2.44}$$

Finally, by (2.42) and (2.43), from (2.44) we obtain the branched continued fraction (2.38) with coefficients defined as (2.39) and (2.40) for ratio (2.25). ■

In this chapter, new three- and four-term recurrence relations of Horn hypergeometric series H_4 are obtained. The expansions of the ratios of these series into formal branched continued fractions are constructed and their special cases are highlighted. The constructed branched continued fractions can be used to establish the domains of analytic continuation of functions, represented by the Horn hypergeometric series H_4 and their ratios.

The results presented in this chapter were published in [57, 66–69].

CHAPTER 3

CONVERGENCE AND ANALYTICAL CONTINUATION

An important application of branched continued fractions is the approximation of analytic functions represented by hypergeometric series. The concept of correspondence plays an important role here. This chapter establishes the domains of analytic continuation of the Horn hypergeometric functions H_4 and their ratios of special cases. These domains will be the domains of convergence of their expansions into branched continued fractions.

3.1. Real parameters cases

First, let us prove the following auxiliary result.

Theorem 3.1. *Let $g_{1,0}$ and $g_{0,k}$, $k \geq 1$, be real numbers such that*

$$0 < g_{1,0} \leq 1,\ 0 < g_{0,k} \leq 1,\ k \geq 1. \tag{3.1}$$

Then:

(A) The branched continued fraction

$$1 - g_{1,0}z_{1,0} - \\ -\cfrac{g_{0,1}z_{0,1}}{1-(1-g_{0,1})z_{1,1}-\cfrac{g_{0,2}(1-g_{0,1})z_{0,2}}{1-(1-g_{0,2})z_{1,2}-\cfrac{g_{0,3}(1-g_{0,2})z_{0,3}}{1-\ddots}}} \tag{3.2}$$

converges for

$$|z_{1,k-1}| \leq \frac{1}{2},\ |z_{0,k}| \leq \frac{1}{2},\ k \geq 1. \tag{3.3}$$

(B) If f_n denotes the nth approximant of (3.2)*, then* $|f_n - 1| \leq 1$, $n \geq 1$.

Proof. We will use the idea of proving Theorem 1 from [91] (see also [92, Theorem 1]). To prove Theorem 3.1(A), we show that the majorant of (3.2), is a branched continued fraction

$$1-g_{1,0}/2-\cfrac{g_{0,1}/2}{1-(1-g_{0,1})/2-\cfrac{g_{0,2}(1-g_{0,1})/2}{1-(1-g_{0,2})/2-\cfrac{g_{0,3}(1-g_{0,2})/2}{1-\ddots}}}. \tag{3.4}$$

Let $F_n^{(n)}=\widehat{F}_n^{(n)}=1$, $n\ge 1$,

$$F_k^{(n)}=1-(1-g_{0,k})z_{1,k}-$$
$$-\cfrac{g_{0,k+1}(1-g_{0,k})z_{0,k+1}}{1-(1-g_{0,k+1})z_{1,k+1}-\cfrac{g_{0,k+2}(1-g_{0,k+1})z_{0,k+2}}{1-\ddots-(1-g_{0,n-1})z_{1,n-1}-g_{0,n}(1-g_{0,n-1})z_{0,n}}}$$

and

$$\widehat{F}_k^{(n)}=1-(1-g_{0,k})/2-$$
$$-\cfrac{g_{0,k+1}(1-g_{0,k})/2}{1-(1-g_{0,k+1})/2-\cfrac{g_{0,k+2}(1-g_{0,k+1})/2}{1-\ddots-(1-g_{0,n-1})/2-g_{0,n}(1-g_{0,n-1})/2}},$$

where $1\le k\le n-1$, $n\ge 2$. Then the following recurrence relations hold

$$F_k^{(n)}=1-(1-g_{0,k})z_{1,k}-\frac{g_{0,k+1}(1-g_{0,k})z_{0,k+1}}{F_{k+1}^{(n)}} \tag{3.5}$$

and

$$\widehat{F}_k^{(n)}=1-(1-g_{0,k})/2-\frac{g_{0,k+1}(1-g_{0,k})/2}{\widehat{F}_{k+1}^{(n)}}, \tag{3.6}$$

where $1\le k\le n-1$, $n\ge 2$. And, thus, for $n\ge 1$ we write the nth approximants of (3.2) and (3.4) as

$$f_n=1-g_{1,0}z_{1,0}-\frac{g_{0,1}z_{0,1}}{F_1^{(n)}},\ \ \widehat{f}_n=1-g_{1,0}/2-\frac{g_{0,1}/2}{\widehat{F}_1^{(n)}},$$

respectively.

Let n be an arbitrary natural number. Using the inequalities (3.1), (3.3), and the recurrence relations (3.5) and (3.6), induction on k, we show the validity of the inequalities

$$F_k^{(n)} \geq \widehat{F}_k^{(n)} \geq g_{0,k}, \ 1 \leq k \leq n, \ n \geq 1. \tag{3.7}$$

For $k = n$ we have

$$F_n^{(n)} = \widehat{F}_n^{(n)} = 1 \geq g_{0,n}.$$

Assuming the validity of (3.7) for $k = r + 1$, where $r + 1 \leq n$, for $k = r$ we obtain

$$|F_r^{(n)}| = \left| 1 - (1 - g_{0,r})z_{1,r} - \frac{g_{0,r+1}(1 - g_{0,r})z_{0,r+1}}{F_{r+1}^{(n)}} \right| \geq$$
$$\geq 1 - (1 - g_{0,r})|z_{1,r}| - \frac{g_{0,r+1}(1 - g_{0,r})|z_{0,r+1}|}{|F_{r+1}^{(n)}|} \geq$$
$$\geq 1 - (1 - g_{0,r})/2 - \frac{g_{0,r+1}(1 - g_{0,r})/2}{\widehat{F}_{r+1}^{(n)}}.$$

Since, by the inequalities (3.3) and (3.7) $\widehat{F}_{r+1}^{(n)} \neq 0$, then, replacing $g_{0,r+1}$ with $\widehat{F}_{r+1}^{(n)}$, we obtain (3.7) for $k = r$.

From this it follows that $F_k^{(n)} \neq 0$ and $\widehat{F}_k^{(n)} \neq 0$, $1 \leq k \leq n$, $n \geq 1$. Applying the formula (1.15), for $n \geq 1$ and $k \geq 1$ we obtain

$$|f_{n+k} - f_n| \leq \frac{g_{0,1}|z_{0,1}| \prod_{r=2}^{n+1} g_{0,r}(1 - g_{0,r-1})|z_{0,r}|}{\prod_{r=1}^{n+1} |F_r^{(n+k)}| \prod_{r=1}^{n} F_r^{(n)}|}$$

$$\leq (-1)^n \frac{(-1)^{n+1}(g_{0,1}/2) \prod_{r=2}^{n+1} g_{0,r}(1 - g_{0,r-1})/2}{\prod_{r=1}^{n+1} \widehat{F}_r^{(n+k)} \prod_{r=1}^{n} \widehat{F}_r^{(n)}} = -(\widehat{f}_{n+k} - \widehat{f}_n). \tag{3.8}$$

It follows that the sequence $\{\widehat{f}_n\}_{n\geq 1}$ is monotonically decreasing. Moreover, by the inequalities (3.7) we have for $n \geq 1$

$$\widehat{f}_n = 1 - g_{1,0}/2 - \frac{g_{0,1}/2}{\widehat{F}_1^{(n)}} \geq 0.$$

That is, the sequence $\{\widehat{f}_n\}_{n\geq 1}$ is bounded from below. Therefore, there exists a limit

$$\widehat{f} = \lim_{n\to+\infty} \widehat{f}_n,$$

which proves Theorem 3.1(A).

Finally, we prove Theorem 3.1(B). By (3.3) and (3.7) for $n \geq 1$ we get

$$|f_n - 1| \leq g_{1,0}|z_{1,0}| + \frac{g_{0,1}|z_{0,1}|}{|F_1^{(n)}|} \leq \frac{1}{2} + \frac{1}{2} = 1$$

and the proof follows. ■

We prove the following theorem.

Theorem 3.2. *Let a and c_1 be real constants such that*

$$c_1 \neq 0,\ u_1 > 0,\ 0 < u_k \leq \tau,\ k \geq 2, \tag{3.9}$$

where u_k, $k \geq 1$, are defined by (2.27)*, τ is a positive number. Then:*

(A) The branched continued fraction (2.26) *converges uniformly on every compact subset of the domain*

$$\mathcal{D}_\tau = \left\{\mathbf{z} \in \mathbb{C}^2 :\ z_k \notin \left[\frac{1}{4(1+\tau)}, +\infty\right),\ k = 1, 2\right\} \tag{3.10}$$

to a function $f(\mathbf{z})$ holomorphic in (3.10)*.*

(B) The function $f(\mathbf{z})$ is an analytic continuation of function (2.23) *in the domain* (3.10)*.*

Proof. To prove Theorem 3.2(A), we will use the idea of proving Theorem 2 from [93] and modify the scheme of its proof.

We set

$$F_n^{(n)}(\mathbf{z}) = 1,\ n \geq 1, \tag{3.11}$$

and

$$F_k^{(n)}(\mathbf{z}) = 1 - z_2 - \cfrac{u_{k+1}z_1}{1 - z_2 - \cfrac{u_{k+2}z_1}{1 - \ddots - z_2 - u_n z_1}}, \tag{3.12}$$

where $1 \leq k \leq n-1,\ n \geq 2$. Then the following relations

$$F_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{u_{k+1}z_1}{F_{k+1}^{(n)}(\mathbf{z})},\ 1 \leq k \leq n-1,\ n \geq 2, \tag{3.13}$$

are valid. Let

$$f_n(\mathbf{z}) = 1 - z_2 - \frac{u_1 z_1}{F_1^{(n)}(\mathbf{z})}$$

be the nth approximant of (2.26), $n \geq 1$. We set

$$\mathcal{D}_{\tau,\alpha} = \left\{ \mathbf{z} \in \mathbb{C}^2 :\ |z_k| + \mathrm{Re}(z_k \mathrm{e}^{-2\mathrm{i}\alpha}) < \frac{\cos^2(\alpha)}{2(1+\tau)},\ k = 1, 2 \right\}, \tag{3.14}$$

where $\alpha \in (-\pi/2, \pi/2)$.

Let n be an arbitrary natural number, α be an arbitrary number from the interval $(-\pi/2, \pi/2)$, and $\mathbf{z}$ be an arbitrary fixed point in (3.14). Using inequalities (3.9), relations (3.11), (3.13), and Lemma 1.1 by induction on k we show that the following inequalities are valid

$$\mathrm{Re}(F_k^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}) > \frac{\cos(\alpha)}{2} > 0,\ 1 \leq k \leq n. \tag{3.15}$$

For $k = n$ we have

$$\mathrm{Re}(F_n^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}) = \cos\alpha > \frac{\cos(\alpha)}{2} > 0.$$

By induction hypothesis that (3.15) hold for $k = r+1$, $r \leq n-1$, we prove (3.15) for $k = r$. Indeed, use of relations (3.13) lead to

$$F_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha} = \mathrm{e}^{-\mathrm{i}\alpha} - \frac{z_2 \mathrm{e}^{-2\mathrm{i}\alpha}}{\mathrm{e}^{-\mathrm{i}\alpha}} - \frac{u_{p+1} z_1 \mathrm{e}^{-2\mathrm{i}\alpha}}{F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}}.$$

We set

$$\xi = -\operatorname{Re}(u_{r+1}z_1\mathrm{e}^{-2\mathrm{i}\alpha}),\ \eta = -\operatorname{Im}(u_{r+1}z_1\mathrm{e}^{-2\mathrm{i}\alpha}),\ u = \operatorname{Re}(F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}),$$
$$v = \operatorname{Im}(F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}).$$

Then it follows from (3.9) and (3.14) that

$$|u_{r+1}z_1\mathrm{e}^{-2\mathrm{i}\alpha}| + \operatorname{Re}(u_{r+1}z_1\mathrm{e}^{-2\mathrm{i}\alpha}) < \frac{\cos^2(\alpha)}{2}.$$

From this inequality it follows that $\eta^2 < 4 - 4\xi$. Now, using Lemma 1.1, inequalities (3.9), (3.14), (3.15), and induction hypothesis, we obtain

$$\operatorname{Re}(F_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}) \geq$$
$$\geq \cos(\alpha) - \frac{|z_2| - \operatorname{Re}(z_2\mathrm{e}^{-2\mathrm{i}\alpha})}{2\operatorname{Re}(\mathrm{e}^{-\mathrm{i}\alpha})} - u_{r+1}\frac{|z_1| - \operatorname{Re}(z_1\mathrm{e}^{-2\mathrm{i}\alpha})}{2\operatorname{Re}(F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha})} >$$
$$> \cos(\alpha) - \frac{\cos(\alpha)}{2(1+\tau)} - u_{r+1}\frac{\cos(\alpha)}{2(1+\tau)} \geq \frac{\cos(\alpha)}{2} > 0,$$

which proves (3.15).

It follows from (3.15) that $F_k^{(n)}(\mathbf{z}) \neq 0$ for all indices. Thus, the approximants of branched continued fraction (2.26) form a sequence of functions holomorphic in the domain (3.14), and, consequently, in (3.10) by virtue of arbitrariness α.

Now, let $\mathcal{K}$ be an arbitrary compact subset of the domain (3.10). Then there exists an open bi-disk

$$\mathcal{O}_\kappa = \{z \in \mathbb{C}^2 :\ |z_k| < \kappa,\ k = 1, 2\},$$

containing $\mathcal{K}$. Let us cover $\mathcal{K}$ with domains of the form $\mathcal{D}_{\tau,\alpha,\kappa} = \mathcal{D}_{\tau,\alpha} \cap \mathcal{O}_\kappa$. From this cover we choose the finite subcover

$$\mathcal{D}_{\tau,\alpha_1,\kappa},\ \mathcal{D}_{\tau,\alpha_1,\kappa},\ \ldots,\ \mathcal{D}_{\tau,\alpha_k,\kappa}.$$

Using (3.9) and (3.15), for the arbitrary $r \in \{1,\ 2,\ \ldots,\ k\}$ we obtain for any $\mathbf{z} \in \mathcal{D}_{\tau,\alpha_r,\kappa}$ and $n \geq 1$

$$|f_n(\mathbf{z})| \leq 1 + |z_2| + \frac{u_1|z_1|}{\operatorname{Re}(F_1^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha_r})} <$$
$$< 1 + \kappa + \frac{2u_1\kappa}{\cos\alpha_r} = C(\mathcal{D}_{\tau,\alpha_r,\kappa}).$$

We set

$$C(\mathcal{K}) = \max_{1\le r\le k} \mathcal{D}_{\tau,\alpha_r,\kappa}.$$

Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|f_n(\mathbf{z})| \le C(\mathcal{K})$, for $n \ge 1$, i.e., the sequence $\{f_n(\mathbf{z})\}_{n\ge1}$ is uniformly bounded on every compact subset of the domain (3.10).

Next, let

$$\chi = \min\left\{\frac{1}{4},\ \frac{1}{4|u_1|},\ \frac{1}{4\tau}\right\}$$

and

$$\mathcal{L}_\chi = \{\mathbf{z} \in \mathbb{R}^2 :\ -\chi < z_k < 0,\ k = 1, 2\}.$$

Then for the arbitrary $\mathbf{z} \in \mathcal{L}_\chi$, $\mathcal{L}_\chi \subset \mathcal{D}_\tau$, we obtain

$$|u_1 z_1| < |u_1|\chi < \frac{1}{4},\ |u_k z_1| < \tau\chi < \frac{1}{4},\ k \ge 2.$$

It follows from Theorem 3.1(A), with $g_{0,k} = 1/2$, $k \ge 1$, that (2.26) converges in the domain $\mathcal{L}_\chi$. Hence, by Theorem 1.4, the branched continued fraction (2.26) converges uniformly on compact subsets of the domain (3.10) to a function $f(z)$ holomorphic in this domain, which proves Theorem 3.2(A).

We prove Theorem 3.2(B) similarly as [49, Theorem 2]. Let

$$E_n^{(n)}(\mathbf{z}) = \frac{H_4(a+n, c_2; c_1+n, c_2; \mathbf{z})}{H_4(a+n+1, c_2; c_1+n+1, c_2; \mathbf{z})},\ n \ge 1, \tag{3.16}$$

where from (2.8), (2.30), and (2.31) it follows that

$$\frac{H_4(a+n, c_2; c_1+n, c_2; \mathbf{z})}{H_4(a+n+1, c_2; c_1+n+1, c_2; \mathbf{z})} = \\ = 1 - z_2 - \cfrac{u_{n+1} z_1}{\cfrac{H_4(a+n+1, c_2; c_1+n+1, c_2; \mathbf{z})}{H_4(a+n+2, c_2; c_1+n+2, c_2; \mathbf{z})}},\ n \ge 1.$$

And let

$$E_k^{(n)}(\mathbf{z}) = 1 - z_2 - \cfrac{u_{k+1} z_1}{1 - z_2 - \cfrac{u_{k+2} z_1}{1 - \ddots - z_2 - \cfrac{u_n z_1}{E_n^{(n)}(\mathbf{z})}}},\ 1 \le k \le n-1,\ n \ge 2.$$

Then

$$E_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{u_{k+1}z_1}{E_{k+1}^{(n)}(\mathbf{z})},\ 1 \le k \le n-1,\ n \ge 2. \tag{3.17}$$

From (2.27), (2.32), (3.16), and (3.17) it follows that for each $n \ge 1$

$$\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})} =$$

$$= 1 - z_2 - \cfrac{u_1 z_1}{1 - z_2 - \cfrac{u_2 z_1}{1 - \ddots - z_2 - \cfrac{u_n z_1}{1 - z_2 - \cfrac{u_{n+1} z_1}{E_n^{(n+1)}(\mathbf{z})}}}} =$$

$$= 1 - z_2 - \frac{u_1 z_1}{E_1^{(n+1)}(\mathbf{z})}.$$

Now, since $F_k^{(n)}(\mathbf{0}) = 1$ and $E_k^{(n)}(\mathbf{0}) = 1$, $1 \le k \le n$ and $n \ge 1$, then for each $1 \le k \le n$ and $n \ge 1$ there exist $\Lambda(1/F_k^{(n)})$ and $\Lambda(1/E_k^{(n)})$, where $\Lambda(\cdot)$ is the Taylor expansion of a function holomorphic in some neighborhood of the origin. In addition, it is clear that $F_k^{(n)}(\mathbf{z}) \not\equiv 0$ and $E_k^{(n)}(\mathbf{z}) \not\equiv 0$ for all indices. Applying the method suggested in [19, p. 28] and relations (3.11), (3.13), and (3.17), for any $n \ge 1$ on the first step we obtain

$$\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})} - f_n(\mathbf{z}) =$$

$$= 1 - z_2 - \frac{u_1 z_1}{E_1^{(n+1)}(\mathbf{z})} - \left(1 - z_2 - \frac{u_1 z_1}{F_1^{(n)}(\mathbf{z})}\right)$$

$$= \frac{u_1 z_1}{E_1^{(n+1)}(\mathbf{z}) F_1^{(n)}(\mathbf{z})} (E_1^{(n+1)}(\mathbf{z}) - F_1^{(n)}(\mathbf{z})).$$

Let k be an arbitrary integer number such that $1 \le k \le n$, $n \ge 1$. Then we have

$$E_k^{(n+1)}(\mathbf{z}) - F_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{u_{k+1} z_1}{E_{k+1}^{(n+1)}(\mathbf{z})} - \left(1 - z_2 - \frac{u_{k+1} z_1}{F_{k+1}^{(n)}(\mathbf{z})}\right) =$$

$$= \frac{u_{k+1} z_1}{E_{k+1}^{(n+1)}(\mathbf{z}) F_{k+1}^{(n)}(\mathbf{z})} (E_{k+1}^{(n+1)}(\mathbf{z}) - F_{k+1}^{(n)}(\mathbf{z})). \tag{3.18}$$

Next, applying recurrence relations (3.18) and taking into account that

$$E_n^{(n+1)}(\mathbf{z}) - F_n^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{u_{n+1}z_1}{E_{n+1}^{(n+1)}(\mathbf{z})} - 1 = -z_2 - \frac{u_{n+1}z_1}{F_{n+1}^{(n+1)}(\mathbf{z})}$$

for any $n \geq 1$ one obtains

$$\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})} - f_n(\mathbf{z}) =$$
$$= -\prod_{r=1}^{n} \frac{u_r z_1}{E_r^{(n+1)}(\mathbf{z}) F_r^{(n)}(\mathbf{z})} \left(z_2 + \frac{u_{n+1}z_1}{E_{n+1}^{(n+1)}(\mathbf{z})} \right). \tag{3.19}$$

From (3.19) it follows that in a neighborhood of zero for any $n \geq 1$ we have

$$\Lambda\left(\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})}\right) - \Lambda(f_n) = \sum_{\substack{p+q \geq n+1 \\ p \geq 0,\ q \geq 0}} \eta_{p,q}^{(n)} z_1^p z_2^q, \tag{3.20}$$

where $\eta_{p,q}^{(n)}$, $p \geq 0$, $q \geq 0$, $p+q \geq n+1$, are some coefficients. From (3.20) it follows that

$$\lambda\left(\Lambda\left(\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})}\right) - \Lambda(f_n)\right) = n+1$$

tends monotonically to $+\infty$ as $n \to +\infty$, i.e., the branched continued fraction (2.26) corresponds (at $\mathbf{z} = \mathbf{0}$) to a formal double power series

$$\Lambda\left(\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})}\right). \tag{3.21}$$

Let $\mathcal{D}$ be the neighborhood of the origin which contained in (3.10) and in which

$$\Lambda\left(\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})}\right) = \sum_{p,q=0}^{+\infty} \eta_{p,q} z_1^p z_2^q.$$

Then, from Theorem 3.2(A) it follows that the sequence $\{f_n(\mathbf{z})\}_{k \geq 1}$ converges uniformly on every compact subset of $\mathcal{D}$ to a function $f(\mathbf{z})$ holomorphic in $\mathcal{D}$. By Theorem 1.2 for arbitrary $p+q$, $p \geq 0$, $q \geq 0$, we have

$$\frac{\partial^{p+q} f_n(\mathbf{z})}{\partial z_1^p \partial z_2^q} \to \frac{\partial^{p+q} f(\mathbf{z})}{\partial z_1^p \partial z_2^p} \text{ as } n \to +\infty$$

on each compact subset of $\mathcal{D}$. Furthermore, according to the above, for each $n \geq 1$ the $\Lambda(f_n)$ and (3.21) agree for all homogeneous terms up to and including degree n. Thus, for any $p+q$, $p \geq 0$, $q \geq 0$, one obtains

$$\lim_{n \to +\infty} \left(\frac{\partial^{p+q} f_n}{\partial z_1^p \partial z_2^q}(\mathbf{0}) \right) = \frac{\partial^{p+q} f}{\partial z_1^p \partial z_2^q}(\mathbf{0}) = p!q!\eta_{p,q}.$$

Hence, for all $\mathbf{z} \in \mathcal{D}$,

$$f(\mathbf{z}) = \sum_{p,q=0}^{+\infty} \left(\frac{\partial^{p+q} f}{\partial z_1^p \partial z_2^q}(\mathbf{0}) \right) \frac{z_1^p}{p!} \frac{z_2^q}{q!} = \sum_{p,q=0}^{+\infty} \eta_{p,q} z_1^p z_2^q.$$

Finally, by Theorem 1.3 and Theorem 3.2(A), Theorem 3.2(B) follows. ■

Remark 3.1. *Conditions* (3.9) *are satisfied when*

$$a > -1, \ c_1 \neq 0, \ 2c_1 > a, \ c_1 > -1.$$

The following result can be proved in much the same way as Theorem 3.2.

Theorem 3.3. *Let a, c_1, c_2 be real constants such that*

$$a > 0, \ c_2 > a, \ v_1 > 0, \ \ 0 < v_k \leq \tau, \ k \geq 2, \tag{3.22}$$

where v_k, $k \geq 1$, are defined by (2.34)*, τ is a positive number. Then:*

(A) The branched continued fraction (2.33) *converges uniformly on every compact subset of the domain* (3.10) *to a function $f(\mathbf{z})$ holomorphic in this domain.*

(B) The function $f(\mathbf{z})$ is an analytic continuation of (2.24) *in* (3.10).

Remark 3.2. *Conditions* (3.22) *are satisfied when*

$$a > 0, \ c_2 > a, \ 2c_1 - a > 1, \ c_1 > 0.$$

Setting $a = 0$ and replacing c_1 by $c_1 - 1$ in Theorem 3.2 or setting $a = 0$ and replacing c_2 by $c_2 - 1$ in Theorem 3.3, we have the following corollary.

Corollary 3.1. *Let* c_1 *be real constant such that*

$$0 < w_k \leq \tau, \ k \geq 1, \tag{3.23}$$

where w_k, $k \geq 1$, *are defined by* (2.37), τ *is a positive number. Then the branched continued fraction* (2.36) *converges uniformly on every compact subset of the domain* (3.10) *to a function* $f(\mathbf{z})$ *holomorphic in this domain, and, in addition, the* $f(\mathbf{z})$ *is an analytic continuation of* (2.35) *in* (3.10).

For the validity of Corollary 3.1, it suffices to show that if $\{f_n(\mathbf{z})\}_{n\geq 1}$ denotes the sequence of approximant of the branched continued fraction (2.36), then

$$f_1(\mathbf{z}) = 1 = F_0^{(0)}(z)$$

and

$$f_n(\mathbf{z}) = \cfrac{1}{1 - z_2 - \cfrac{w_1 z_1}{1 - z_2 - \cfrac{w_2 z_1}{1 - \ddots - z_2 - w_{n-1} z_1}}} =$$

$$= \frac{1}{F_0^{(n-1)}(\mathbf{z})}, \ n \geq 2,$$

By analogy to prove the inequalities (3.15) it can be shown that

$$\mathrm{Re}(F_0^{(n-1)}(\mathbf{z})) > \frac{\cos(\alpha)}{2} > 0, \ n \geq 1, \ \mathbf{z} \in \mathcal{D}_\tau,$$

where $\mathcal{D}_\tau$ is defined by (3.10). Hence $\{f_n(\mathbf{z})\}_{n\geq 1}$ is a sequence of functions holomorphic in the domain (3.10).

Remark 3.3. *Conditions* (3.23) *are satisfied when* $2c_1 > 1$.

Now, we prove the following theorem.

Theorem 3.4. *Let* a, c_1, *and* c_2 *be real constants such that*

$$d_0 \neq 0, \ d_1 > 0, \ h_1 > 0, \ d_1 + h_1 \leq 1 + \tau, \ 0 < h_k \leq \tau, \ k \geq 2, \tag{3.24}$$

where τ *is a positive number,* d_0, d_1, h_1 *and* h_k, $k \geq 2$, *are defined in* (2.39) *and* (2.40), *respectively. Then:*

(A) The branched continued fraction (2.38) *converges uniformly on every compact subset of the domain* (3.10) *to a function* $f(\mathbf{z})$ *holomorphic in this domain.*

(B) The function $f(\mathbf{z})$ *is an analytic continuation of function* (2.25) *in the domain* (3.10)*.*

Proof. To prove Theorem 3.4, in much the same way as Theorem 3.2. We set

$$W_n^{(n)}(\mathbf{z}) = 1, \; n \geq 1, \tag{3.25}$$

and

$$W_k^{(n)}(\mathbf{z}) = 1 - d_k z_2 - \cfrac{h_k z_1}{1 - d_{k+1} z_2 - \cfrac{h_{k+1} z_1}{1 - \ddots - d_{n-1} z_2 - h_{n-1} z_1}}, \tag{3.26}$$

where $1 \leq k \leq n-1$, $n \geq 2$. Then

$$W_k^{(n)}(\mathbf{z}) = 1 - d_k z_2 - \frac{h_k z_1}{W_{k+1}^{(n)}(\mathbf{z})}, \; 1 \leq k \leq n-1, \; n \geq 2, \tag{3.27}$$

Let

$$f_n(\mathbf{z}) = 1 + \frac{d_0 z_2}{W_1^{(n)}(\mathbf{z})}$$

be the nth approximant of (2.38), $n \geq 1$.

Let n be an arbitrary natural number, α be an arbitrary number from the interval $(-\pi/2, \pi/2)$, and $\mathbf{z}$ be an arbitrary fixed point in (3.14). By induction on k we show that the following inequalities are valid

$$\mathrm{Re}(W_k^{(n)}(\mathbf{z}) \mathrm{e}^{-\mathrm{i}\alpha}) > \frac{\cos(\alpha)}{2} > 0, \; 1 \leq k \leq n. \tag{3.28}$$

For $k = n$ the inequalities are obvious. By induction hypothesis that (3.28) hold for $k = r+1$, $r \leq n-1$, we prove (3.28) for $k = r$. Indeed, use of relations

(3.27) and Lemma 1.1 lead to

$$
\begin{aligned}
&\operatorname{Re}(W_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}) = \\
&= \operatorname{Re}(\mathrm{e}^{-\mathrm{i}\alpha}) - \operatorname{Re}\left(\frac{d_r z_2 \mathrm{e}^{-2\mathrm{i}\alpha}}{\mathrm{e}^{-\mathrm{i}\alpha}}\right) - \operatorname{Re}\left(\frac{h_r z_1 \mathrm{e}^{-2\mathrm{i}\alpha}}{W_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}}\right) \geq \\
&\geq \cos(\alpha) - d_r \frac{|z_2| - \operatorname{Re}(z_2 \mathrm{e}^{-2\mathrm{i}\alpha})}{2\operatorname{Re}(\mathrm{e}^{-\mathrm{i}\alpha})} - h_r \frac{|z_1| - \operatorname{Re}(z_1 \mathrm{e}^{-2\mathrm{i}\alpha})}{2\operatorname{Re}(W_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha})} > \\
&> \cos(\alpha) - d_r \frac{\cos(\alpha)}{2(1+\tau)} - h_r \frac{\cos(\alpha)}{2(1+\tau)} = \\
&= \cos(\alpha) - \frac{\cos(\alpha)}{2(1+\tau)}(d_r + h_r) \geq \frac{\cos(\alpha)}{2} > 0,
\end{aligned}
$$

which proves (3.28).

It follows from (3.28) that $W_k^{(n)}(\mathbf{z}) \neq 0$ for all indices. Thus, the approximants of branched continued fraction (2.38) form a sequence of functions holomorphic in the domain (3.14), and, consequently, in (3.10) by virtue of arbitrariness α.

Now, let $\mathcal{K}$ be an arbitrary compact subset of the domain (3.10). Then there exists an open bi-disk

$$\mathcal{O}_\kappa = \{z \in \mathbb{C}^2 : \ |z_k| < \kappa, \ k = 1, 2\},$$

containing $\mathcal{K}$. Let us cover $\mathcal{K}$ with domains of the form

$$\mathcal{D}_{\tau,\alpha,\kappa} = \mathcal{D}_{\tau,\alpha} \cap \mathcal{O}_\kappa,$$

where $\mathcal{D}_{\tau,\alpha}$ is defined by (3.14). From this cover we choose the finite subcover

$$\mathcal{D}_{\tau,\alpha_1,\kappa}, \ \mathcal{D}_{\tau,\alpha_1,\kappa}, \ \ldots, \ \mathcal{D}_{\tau,\alpha_k,\kappa}.$$

Using (3.15) and (3.24), for the arbitrary $r \in \{1, \ 2, \ \ldots, \ k\}$ we obtain for any $\mathbf{z} \in \mathcal{D}_{\tau,\alpha_r,\kappa}$ and $n \geq 1$

$$
\begin{aligned}
|f_n(\mathbf{z})| &\leq 1 + \frac{|d_0||z_2|}{\operatorname{Re}(W_1^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha_r})} < \\
&< 1 + \frac{2|d_0|\kappa}{\cos\alpha_r} = C(\mathcal{D}_{\tau,\alpha_r,\kappa}).
\end{aligned}
$$

We set

$$C(\mathcal{K}) = \max_{1\le r\le k} \mathcal{D}_{\tau,\alpha_r,\kappa}.$$

Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|f_n(\mathbf{z})| \le C(\mathcal{K})$, for $n \ge 1$, i.e., the sequence $\{f_n(\mathbf{z})\}_{n\ge1}$ is uniformly bounded on every compact subset of the domain (3.10).

Next, let

$$\chi = \min\left\{\frac{1}{4},\ \frac{1}{4\tau},\ \frac{1}{4|d_1|},\ \frac{1}{4|h_1|}\right\}$$

and

$$\mathcal{L}_\chi = \{\mathbf{z} \in \mathbb{R}^2 :\ -\chi < z_k < 0,\ k = 1, 2\}.$$

Then for the arbitrary $\mathbf{z} \in \mathcal{L}_\chi$, $\mathcal{L}_\chi \subset \mathcal{D}_\tau$, we obtain

$$|d_1 z_1| < |d_1|\chi < \frac{1}{4},\ |h_1 z_1| < |h_1|\chi < \frac{1}{4},\ |h_k z_1| < \tau\chi < \frac{1}{4},\ k \ge 2.$$

It follows from Theorem 3.1(A), with $g_{0,k} = 1/2$, $k \ge 1$, that branched continued fraction (2.38) converges in the domain $\mathcal{L}_\chi$. Hence, by Theorem 1.4, the branched continued fraction (2.38) converges uniformly on compact subsets of the domain (3.10) to a function $f(z)$ holomorphic in this domain, which proves Theorem 3.4(A).

Next, we prove Theorem 3.4(B). Let

$$V_1^{(1)}(\mathbf{z}) = \frac{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})}{H_4(a+1, c_2+2; c_1, c_2+2; \mathbf{z})},$$

$$V_n^{(n)}(\mathbf{z}) = \frac{H_4(a+n-1, c_2+2; c_1+n-2, c_2+2; \mathbf{z})}{H_4(a+n, c_2+2; c_1+n-1, c_2+2; \mathbf{z})},\ n \ge 2,$$

and

$$V_k^{(n)}(\mathbf{z}) = 1 - d_k z_2 - \cfrac{h_k z_1}{1 - d_{k+1} z_2 - \cfrac{h_{k+1} z_1}{1 - \ddots - d_{n-1} z_2 - \cfrac{h_{n-1} z_1}{V_n^{(n)}(\mathbf{z})}}},$$

where $1 \le k \le n-1$, $n \ge 2$. Then

$$V_k^{(n)}(\mathbf{z}) = 1 - d_k z_2 - \frac{h_k z_1}{V_{k+1}^{(n)}(\mathbf{z})},\ 1 \le k \le n-1,\ n \ge 2. \tag{3.29}$$

Thus, for each $n \geq 1$ we have the following

$$\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})} =$$

$$= 1 + \cfrac{d_0 z_2}{1 - d_1 z_2 - \cfrac{h_1 z_1}{1 - d_2 z_2 - \cfrac{h_2 z_1}{1 - \ddots - d_{n-1} z_2 - \cfrac{h_{n-1} z_1}{V_n^{(n+1)}(\mathbf{z})}}}} =$$

$$= 1 + \frac{d_0 z_2}{V_1^{(n+1)}(\mathbf{z})}.$$

Let the mapping $\Lambda : f(\mathbf{z}) \to \Lambda(f)$ associate with $f(\mathbf{z})$ its Taylor expansion in a neighbourhood of the origin. Let n be an arbitrary natural number. Since $W_k^{(n)}(\mathbf{0}) = 1$ and $V_k^{(n)}(\mathbf{0}) = 1$, $1 \leq k \leq n$, $n \geq 1$, then for each $1 \leq k \leq n$ and $n \geq 1$ there exist $\Lambda(1/W_k^{(n)})$ and $\Lambda(1/V_k^{(n)})$. In addition, it is clear that $W_k^{(n)}(\mathbf{z}) \not\equiv 0$ and $V_k^{(r)}(\mathbf{z}) \not\equiv 0$, where $1 \leq k \leq n$ and $n \geq 1$.

Using recurrence relations (3.27), (3.29) and the idea of constructing the difference of two approximants of a branched continued fraction [19, p. 28], for $n \geq 1$ on the first step we have

$$\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})} - f_n(\mathbf{z})$$

$$== 1 + \frac{d_0 z_2}{V_1^{(n+1)}(\mathbf{z})} - \left(1 + \frac{d_0 z_2}{W_1^{(n)}(\mathbf{z})}\right) =$$

$$= -\frac{d_0 z_2}{V_1^{(n+1)}(\mathbf{z}) W_1^{(n)}(\mathbf{z})} (V_1^{(n+1)}(\mathbf{z}) - W_1^{(n)}(\mathbf{z})).$$

Let k be an arbitrary integer number that satisfy the inequalities $1 \leq k \leq n-1$ and $n \geq 2$. Then

$$V_k^{(n+1)}(\mathbf{z}) - W_k^{(n)}(\mathbf{z}) =$$

$$= 1 - d_k z_2 - \frac{h_k z_1}{V_{k+1}^{(n+1)}(\mathbf{z})} - \left(1 - d_k z_2 - \frac{h_k z_1}{W_{k+1}^{(n)}(\mathbf{z})}\right) =$$

$$= \frac{h_k z_1}{V_{k+1}^{(n+1)}(\mathbf{z}) W_{k+1}^{(n)}(\mathbf{z})} (V_{k+1}^{(n+1)}(\mathbf{z}) - W_{k+1}^{(n)}(\mathbf{z})). \qquad (3.30)$$

Since

$$V_n^{(n+1)}(\mathbf{z}) - W_n^{(n)}(\mathbf{z}) =$$
$$= 1 - d_n z_2 - \frac{h_n z_1}{V_{n+1}^{(n+1)}(\mathbf{z})} - 1 = -d_n z_2 - \frac{h_n z_1}{V_{n+1}^{(n+1)}(\mathbf{z})},$$

then by recurrence relations (3.30) on the nth step we obtain

$$\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})} - f_n(\mathbf{z}) =$$
$$= d_0 z_1^{n-1} z_2 \left(z_2 + \frac{h_n z_1}{V_{n+1}^{(n+1)}(\mathbf{z})} \right) \prod_{r=1}^{n-1} \frac{h_r}{V_r^{(n+1)}(\mathbf{z}) W_r^{(n)}(\mathbf{z})}. \tag{3.31}$$

Thus, from (3.31) in a neighborhood of origin for any $n \geq 1$ we get

$$\Lambda\left(\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})}\right) - \Lambda(f_n) = \sum_{\substack{p+q \geq n+1 \\ p \geq 0,\ q \geq 0}} \eta_{p,q}^{(n)} z_1^p z_2^q, \tag{3.32}$$

where $\eta_{p,q}^{(n)}$, $p \geq 0$, $q \geq 0$, $p+q \geq n+1$, are some coefficients.

From (3.32) it follows that

$$\lambda\left(\Lambda\left(\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})}\right) - \Lambda(f_n)\right) = n+1$$

tends monotonically to $+\infty$ as $n \to +\infty$,. This means that the branched continued fraction (2.38) corresponds to a

$$\Lambda\left(\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})}\right) \tag{3.33}$$

at $\mathbf{z} = \mathbf{0}$.

Let $\mathcal{D}$ be the neighborhood of the origin such that $\mathcal{D} \subset \mathcal{D}_\tau$, where $\mathcal{D}_\tau$ is defined by (3.10), and such that

$$\Lambda\left(\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})}\right) = \sum_{\substack{p+q \geq 0 \\ p \geq 0,\ q \geq 0}} \eta_{p,q} z_1^p z_2^q.$$

Then, from above proof it follows that the branched continued fraction (2.38) converges uniformly on every compact subset of $\mathcal{D}$ to a function $f(\mathbf{z})$ holomorphic in the domain $\mathcal{D}$. Next, by Theorem 1.2 for arbitrary $p+q$, $p \geq 0$, $q \geq 0$, we get

$$\frac{\partial^{p+q} f_n(\mathbf{z})}{\partial z_1^p \partial z_2^q} \to \frac{\partial^{p+q} f(\mathbf{z})}{\partial z_1^p \partial z_2^q}$$

as $n \to +\infty$ on each compact subset of $\mathcal{D}$. In addition, as in the above proof, for each $n \geq 1$ the $\Lambda(f_n)$ and (3.33) agree for all homogeneous terms up to and including degree n. Thus, for an arbitrary $p+q$ such that $p \geq 0$, $q \geq 0$, we have

$$\lim_{n \to +\infty} \left(\frac{\partial^{p+q} f_n}{\partial z_1^p \partial z_2^q}(\mathbf{0}) \right) = \frac{\partial^{p+q} f}{\partial z_1^p \partial z_2^q}(\mathbf{0}) = p!q!\eta_{p,q}.$$

Hence, for all $\mathcal{D}$,

$$f(\mathbf{z}) = \sum_{\substack{p+q \geq 0 \\ p \geq 0,\ q \geq 0}} \left(\frac{\partial^{p+q} f}{\partial z_1^p \partial z_2^q}(\mathbf{0}) \right) \frac{z_1^p}{p!} \frac{z_2^q}{q!} = \sum_{\substack{p+q \geq 0 \\ p \geq 0,\ q \geq 0}} \eta_{p,q} z_1^p z_2^q.$$

Finally, according to Theorem 1.3 and Theorem 3.4(A), Theorem 3.4(B) follows. ■

Remark 3.4. *Conditions* (3.24) *are satisfied when* $0 < a < c_2 + 1$ *and* $a + 1 < 2c_1$.

Next, we prove the following theorem.

Theorem 3.5. *Let* a *and* c_1 *be real constants satisfying the inequalities* (3.9), *where* u_k, $k \geq 1$, *are defined by* (2.27), τ *is a positive number. Then:*

(A) The branched continued fraction (2.26) *converges uniformly on every compact subset of the domain*

$$\mathcal{P}_\tau = \left\{ \mathbf{z} \in \mathbb{C}^2 :\ z_1 \notin \left[\frac{1}{8\tau}, +\infty \right),\ z_2 \notin \left[\frac{1}{4}, +\infty \right) \right\} \tag{3.34}$$

to a function $f(\mathbf{z})$ *holomorphic in* (3.34).

(B) The function $f(\mathbf{z})$ *is an analytic continuation of function* (2.23) *in the domain* (3.34).

Proof. As in the proof of Theorem 3.2, we show that the inequalities (3.15) is valid. We set

$$\mathcal{P}_{\tau,\alpha}=\Big\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|+\mathrm{Re}(z_1\mathrm{e}^{-2\mathrm{i}\alpha})<\frac{\cos^2(\alpha)}{4\tau},$$

$$\mathrm{Re}(z_2\mathrm{e}^{-\mathrm{i}\alpha})<\frac{\cos(\alpha)}{4}\Big\},\qquad(3.35)$$

where $\alpha\in(-\pi/2,\pi/2)$.

Let n be an arbitrary natural number, α be an arbitrary number from the interval $(-\pi/2,\pi/2)$, and $\mathbf{z}$ be an arbitrary fixed point in $\mathcal{P}_{\tau,\alpha}$. From (3.11) it is clear that for $k=n$ the inequalities (3.15) hold. By induction hypothesis that (3.15) hold for $k=r+1$, $r\le n-1$, we prove (3.15) for $k=r$. Indeed, use of Lemma 1.1, relations (3.13), and inequalities (3.9), (3.13)–(3.15), lead to

$$\mathrm{Re}(F_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha})=$$

$$=\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha})-\mathrm{Re}(z_2\mathrm{e}^{-\mathrm{i}\alpha})-\mathrm{Re}\left(\frac{u_{r+1}z_1\mathrm{e}^{-2\mathrm{i}\alpha}}{F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha}}\right)\ge$$

$$\ge\cos(\alpha)-\mathrm{Re}(z_2\mathrm{e}^{-\mathrm{i}\alpha})-u_{r+1}\frac{|z_1|-\mathrm{Re}(z_1\mathrm{e}^{-2\mathrm{i}\alpha})}{2\mathrm{Re}(F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha})}>$$

$$>\cos(\alpha)-\frac{\cos(\alpha)}{4}-u_{r+1}\frac{\cos(\alpha)}{4\tau}\ge\frac{\cos(\alpha)}{2}>0,$$

which proves (3.15).

It follows from (3.15) that $F_k^{(n)}(\mathbf{z})\neq 0$ for all indices. Thus, the approximants of (2.26) form a sequence of functions holomorphic in the domain (3.35), and, consequently, in (3.34) by virtue of arbitrariness α.

Now, let $\mathcal{K}$ be an arbitrary compact subset of the domain (3.34), and $f_n(\mathbf{z})$, $n\ge 1$, be approximants of the branched continued fraction (2.26). Then there exists an open bi-disk

$$\mathcal{O}_\kappa=\{z\in\mathbb{C}^2:\ |z_k|<\kappa,\ k=1,2\},$$

containing $\mathcal{K}$. Let us cover $\mathcal{K}$ with domains of the form

$$\mathcal{P}_{\tau,\alpha,\kappa}=\mathcal{P}_{\tau,\alpha}\cap\mathcal{O}_\kappa,$$

where $\mathcal{P}_{\tau,\alpha}$ is defined by (3.35). From this cover we choose the finite subcover

$$\mathcal{P}_{\tau,\alpha_1,\kappa},\ \mathcal{P}_{\tau,\alpha_1,\kappa},\ \ldots,\ \mathcal{P}_{\tau,\alpha_k,\kappa}.$$

Using (3.9) and (3.15), for the arbitrary $r \in \{1,\ 2,\ \ldots,\ k\}$ we obtain for any $\mathbf{z} \in \mathcal{P}_{\tau,\alpha_r,\kappa}$ and $n \geq 1$

$$|f_n(\mathbf{z})| \leq 1 + |z_2| + \frac{u_1|z_1|}{\mathrm{Re}(F_1^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\alpha_r})} < 1 + \kappa + \frac{2u_1\kappa}{\cos\alpha_r} = C(\mathcal{P}_{\tau,\alpha_r,\kappa}).$$

We set

$$C(\mathcal{K}) = \max_{1\leq r\leq k} \mathcal{P}_{\tau,\alpha_r,\kappa}.$$

Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|f_n(\mathbf{z})| \leq C(\mathcal{K})$, for $n \geq 1$, i.e., the sequence $\{f_n(\mathbf{z})\}_{n\geq 1}$ is uniformly bounded on every compact subset of the domain (3.34).

Next, let

$$\chi = \min\left\{\frac{1}{4},\ \frac{1}{4u_1},\ \frac{1}{4\tau}\right\}$$

and

$$\mathcal{L}_\chi = \{\mathbf{z} \in \mathbb{R}^2 :\ -\chi < z_k < 0,\ k = 1, 2\}.$$

Then for the arbitrary $\mathbf{z} \in \mathcal{L}_\chi$, $\mathcal{L}_\chi \subset \mathcal{P}_\tau$, we obtain

$$|u_1 z_1| < u_1\chi < \frac{1}{4},\ |u_k z_1| < \tau\chi < \frac{1}{4},\ k \geq 2.$$

It follows from Theorem 3.1(A), with $g_{0,k} = 1/2$, $k \geq 1$, that (2.26) converges in the domain $\mathcal{L}_\chi$. Hence, by Theorem 1.4, the branched continued fraction (2.26) converges uniformly on compact subsets of the domain (3.34) to a function $f(z)$ holomorphic in this domain, which proves Theorem 3.5(A).

Finally, from Theorem 3.2(B) it follows that the branched continued fraction (2.26) corresponds at $\mathbf{z} = \mathbf{0}$ to the function (2.23). Therefore, according to Theorem 1.3 and Theorem 3.5(A), Theorem 3.5(B) follows. ■

We consider the following example.

Example 3.1. *From [89] and Theorem 3.5 it follows*

$$\frac{1-z_2+\sqrt{(1-z_2)^2-4z_1}}{2}=\frac{H_4(-1/2,c_2;1/2,c_2;\mathbf{z})}{H_4(1/2,c_2;3/2,c_2;\mathbf{z})}=$$
$$=1-z_2-\cfrac{z_1}{1-z_2-\cfrac{z_1}{1-z_2-\ddots}},\tag{3.36}$$

where the branched continued fraction converges and represents a single-valued branch of the analytic function

$$\frac{1-z_2+\sqrt{(1-z_2)^2-4z_1}}{2}\tag{3.37}$$

in the domain

$$\mathcal{P}=\left\{\mathbf{z}\in\mathbb{C}^2:\ z_1\notin\left[\frac{1}{8},+\infty\right),\ z_2\notin\left[\frac{1}{4},+\infty\right)\right\}.\tag{3.38}$$

The following result can be proved in much the same way as Theorem 3.5.

Theorem 3.6. *Let a, c_1, c_2 be real constants satisfying the inequalities* (3.22)*, where v_k, $k\ge 1$, are defined by* (2.34)*, τ is a positive number. Then the branched continued fraction* (2.33) *converges uniformly on every compact subset of the domain* (3.34) *to a function $f(\mathbf{z})$ holomorphic in this domain, and, in addition, the function $f(\mathbf{z})$ is an analytic continuation of* (2.24) *in the domain* (3.34).

Setting $a=0$ and replacing c_1 by c_1-1 in Theorem 3.5 or setting $a=0$ and replacing c_2 by c_2-1 in Theorem 3.6, we have the following corollary.

Corollary 3.2. *Let c_1 be real constant satisfying the inequalities* (3.23)*, where w_k, $k\ge 1$, are defined by* (2.37)*, τ is a positive number. Then the branched continued fraction* (2.36) *converges uniformly on every compact subset of the domain* (3.34) *to a function $f(\mathbf{z})$ holomorphic in this domain, and, in addition, the $f(\mathbf{z})$ is an analytic continuation of the function* (2.35) *in* (3.34).

We consider the following example.

Example 3.2. *From [89] and Corollary 3.2 it follows that*

$$
\begin{aligned}
&((1-z_2)^2-4z_1)^{-1/2} = H_4(1,b;1,b;\mathbf{z}) = \\
&= \cfrac{1}{1-z_2-\cfrac{2z_1}{1-z_2-\cfrac{z_1}{1-z_2-\cfrac{z_1}{1-\ddots}}}}
\end{aligned} \tag{3.39}
$$

and, in addition, the branched continued fraction in (3.39) *converges and represents a single-valued branch of the analytic function of two variables*

$$((1-z_2)^2-4z_1)^{-1/2} \tag{3.40}$$

in the domain (3.38).

Let us consider another example.

Example 3.3. *From [89] and Corollary 3.2 it follows that*

$$
\begin{aligned}
&\arctan\frac{2\sqrt{-z_1}}{1-z_2} = 2\sqrt{-z_1}H_4(1,b;3/2,b;\mathbf{z}) = \\
&= \cfrac{2\sqrt{-z_1}}{1-z_2-\cfrac{\frac{4}{3}z_1}{1-z_2-\cfrac{\frac{16}{15}z_1}{1-\ddots-z_2-\cfrac{\frac{k^2}{k^2-1/4}z_1}{1-\ddots}}}},
\end{aligned} \tag{3.41}
$$

where the branched continued fraction converges and represents a single-valued branch of the analytic function of two variables

$$\arctan\frac{2\sqrt{-z_1}}{1-z_2} \tag{3.42}$$

in the domain

$$\mathcal{T} = \left\{\mathbf{z}\in\mathbb{C}^2:\ z_1\notin[0,+\infty),\ z_2\notin\left[\frac{1}{4},+\infty\right)\right\}. \tag{3.43}$$

Finally, the following result can also be proved in much the same way as Theorem 3.5.

Theorem 3.7. *Let a, c_1, and c_2 be real constants such that*

$$c_2 \notin \{-1, 0\},\ d_0 > 0,\ d_1 > 0,\ 0 < h_k \le \tau,\ k \ge 1, \tag{3.44}$$

where τ is a positive number, d_0, d_1, h_1 and h_k, $k \ge 2$, are defined in (2.39) *and* (2.40)*, respectively. Then the branched continued fraction* (2.38) *converges uniformly on every compact subset of the domain*

$$\mathcal{P}_\tau = \left\{ \mathbf{z} \in \mathbb{C}^2 :\ z_1 \notin \left[\frac{1}{8\tau}, +\infty\right),\ z_2 \notin \left[\frac{1}{4\max\{1, d_1\}}, +\infty\right) \right\} \tag{3.45}$$

to a function $f(\mathbf{z})$ holomorphic in this domain, and, in addition, the function $f(\mathbf{z})$ is an analytic continuation of (2.25) *in the domain $\mathcal{P}_\tau$.*

Remark 3.5. *In Theorems 3.2–3.7 and Corollaries 3.1 and 3.2, the assumption on the sequence coefficients* (2.27), (2.34), (2.37), (2.39), *and* (2.40) *(involves (together with positivity) an upper bound τ, and that the domains of the analytic continuation also depends on this τ; and the smaller τ, the larger domains.*

3.2. Complex parameters cases

Let us prove the following theorem.

Theorem 3.8. *Let a and c_1 be complex constants such that*

$$|u_k| + \mathrm{Re}(u_k) \le \mu\nu(1 - \nu),\ k \ge 2, \tag{3.46}$$

where u_k, $k \ge 2$, are defined by (2.27)*, $c_1 \notin \{0, -1, -2, \ldots\}$, μ is a positive number, and $0 < \nu < 1$. Then:*

(A) The branched continued fraction (2.26) *converges uniformly on every compact subset of the domain*

$$\mathcal{H}_{\mu,\nu}^{\kappa,\tau} = \mathcal{H}_{\mu,\nu} \cup \mathcal{H}^{\kappa,\tau} \tag{3.47}$$

to a function $f(\mathbf{z})$ *holomorphic in* (3.47), *where*

$$\mathcal{H}_{\mu,\nu} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{1 + \cos(\arg(z_1))}{2\mu}, \right.$$
$$\left. \operatorname{Re}(z_2 \mathrm{e}^{-(\mathrm{i}/2)\arg(z_1)}) < \frac{\nu}{2} \cos\left(\frac{\arg(z_1)}{2} \right) \right\} \tag{3.48}$$

and

$$\mathcal{H}^{\kappa,\tau} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{\kappa(1-\kappa)}{2\tau}, \ |z_2| < \frac{1-\kappa}{2} \right\}, \tag{3.49}$$

with

$$\tau = \max_{k \ge 2}\{|u_k|\}, \ 0 < \kappa < 1. \tag{3.50}$$

(B) The function $f(\mathbf{z})$ *is an analytic continuation of* (2.23) *in* (3.47).

Proof. An application of Theorem 3.1(A), with $g_{1,0} = 1$ and

$$g_{0,k} = \kappa, \ k \ge 1, \ 0 < \kappa < 1,$$

shows that the branched continued fraction (2.26) converges for all $\mathbf{z} \in \mathcal{H}^{\kappa,\tau}$. Theorem 3.1(B) implies that the approximants of branched continued fraction (2.26) all lie in $|w-1| \le 1$ if $\mathbf{z} \in \mathcal{H}^{\kappa,\tau}$. Hence, by Theorem 1.4, the convergence of branched continued fraction (2.26) is uniform on every compact subset of the domain (3.49).

Let us prove the uniform convergence of the branched continued fraction (2.26) on every compact subset of the domain (3.48). Let n be an arbitrary natural number, $\arg(z_1) = \varphi$, and $\mathbf{z}$ be an arbitrary fixed point in the domain (3.48). Let us prove that

$$\operatorname{Re}(F_k^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}) > (1-\nu)\cos(\varphi/2) \ge c > 0, \ 1 \le k \le n, \ n \ge 1, \tag{3.51}$$

where $F_k^{(n)}(\mathbf{z})$, $1 \le k \le n$, $n \ge 1$, are defined by (3.11) and (3.12).

From an arbitrary fixed point $\mathbf{z}$, $\mathbf{z} \in \mathcal{H}_{\mu,\nu}$, it follows that for its arbitrary neighborhood, there exists ε, $0 < \varepsilon \le \pi/2$, such that $|\varphi/2| \le \pi/2 - \varepsilon$ and, thus,

$$(1-\nu)\cos(\varphi/2) \ge (1-\nu)\cos(\varphi/2 - \sigma) = (1-\nu)\sin(\varepsilon) = c > 0.$$

If $k=n$, then

$$\mathrm{Re}(F_n^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})=\cos(\varphi/2)>(1-\nu)\cos(\varphi/2).$$

Assuming that the first inequality in (3.51) is true if $k=r+1\le n$. Then, for $k=r$ from (3.13) we obtain

$$F_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}=\mathrm{e}^{-\mathrm{i}\varphi/2}-z_2\mathrm{e}^{-\mathrm{i}\varphi/2}-\frac{u_{r+1}z_1\mathrm{e}^{-\mathrm{i}\varphi}}{F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}}. \tag{3.52}$$

Using Lemma 1.1, (3.46), (3.48), and (3.51), from (3.52) we get

$$\begin{gathered}\mathrm{Re}(F_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})=\\ =\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\varphi/2})-\mathrm{Re}(z_2\mathrm{e}^{-\mathrm{i}\varphi/2})-\mathrm{Re}\left(\frac{u_{r+1}z_1\mathrm{e}^{-\mathrm{i}\varphi}}{F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}}\right)\ge\\ \ge\cos(\varphi/2)-\mathrm{Re}(z_2\mathrm{e}^{-\mathrm{i}\varphi/2})-\frac{|u_{r+1}|+\mathrm{Re}(u_{r+1})}{2\,\mathrm{Re}(F_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})}|z_1|>\\ >\cos(\varphi/2)-\frac{\nu\cos(\varphi/2)}{2}-\frac{\mu\nu(1-\nu)}{2(1-\nu)\cos(\varphi/2)}\frac{1+\cos(\varphi)}{2\mu}=\\ =(1-\nu)\cos(\varphi/2).\end{gathered}$$

Thus, $F_1^{(n)}(\mathbf{z})\ne 0$, $n\ge 1$, and $\mathbf{z}\in\mathcal{H}_{\mu,\nu}$, i.e. that each approximant of branched continued fraction (2.26) is a holomorphic function in (3.48).

Now, let $\mathcal{K}$ be an arbitrary compact subset of (3.48), and $f_n(\mathbf{z})$, $n\ge 1$, be approximants of the branched continued fraction (2.26), then there exists an open bi-disk

$$\mathcal{O}_\eta=\{\mathbf{z}\in\mathbb{C}^2:\ |z_k|<\eta,\ k=1,2\}$$

of radius η, $\eta>0$, such that $\mathcal{K}\subset\mathcal{O}_\eta$. Then, for any $n\ge 1$ and

$$\mathbf{z}\in\mathcal{H}_{\mu,\nu}\cap\mathrm{O}_\eta$$

from we get

$$\begin{gathered}|f_n(\mathbf{z})|\le 1+\eta+\frac{|u_1|\eta}{\mathrm{Re}(F_1^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})}<\\ <1+\eta+\frac{\tau\eta}{(1-\nu)\cos(\varphi/2)}=C(\mathcal{K}),\end{gathered}$$

i.e. the sequence $\{f_n(\mathbf{z})\}_{\geq 1}$ is uniformly bounded on every compact subset of the domain $\mathcal{H}_{\mu,\nu}$.

It is clear that for every χ such that

$$0 < \chi < \min\left\{\frac{1-\kappa}{2}, \frac{\kappa(1-\kappa)}{2\tau}, \frac{1}{\mu}, \frac{\nu}{2}\right\}$$

the domain

$$\mathcal{L}_\chi = \{\mathbf{z} \in \mathbb{R}^2 : \ 0 < z_k < \chi, \ k = 1, 2\}$$

contained in $\mathcal{H}_{\mu,\nu}$, in particular $\mathcal{L}_{\chi/2} \subset \mathcal{H}_{\mu,\nu}$. Using (3.50), for any $\mathbf{z} \in \mathcal{L}_\chi$, $\mathcal{L}_\chi \subset \mathcal{H}_{\mu,\nu}$, we have

$$|z_2| < \frac{1-\kappa}{2}, \ |u_k z_1| < \frac{\kappa(1-\kappa)}{2}, \ k \geq 2,$$

i.e. the elements of branched continued fraction (2.26) satisfy Theorem 3.1, with

$$g_{1,0} = 1, \ g_{0,k} = \kappa, \ k \geq 1.$$

It shows that branched continued fraction (2.26) converges for all $\mathbf{z} \in \mathcal{L}_\chi$, $\mathcal{L}_\chi \subset \mathcal{H}_{\mu,\nu}$. Therefore, by Theorem 1.4, the convergence of the branched continued fraction (2.26) is uniform on compact subsets of the domain (3.48), and hence (3.47), which proves Theorem 3.8(A).

Finally, Theorem 3.8(B) follows from Theorem 1.3, Theorem 3.2(B), and Theorem 3.8(A). ■

The following theorem can be proved in much the same way as Theorem 3.8.

Theorem 3.9. *Let a, c_1, and c_2 be complex constants such that*

$$|v_k| + \mathrm{Re}(v_k) \leq \mu\nu(1-\nu), \ k \geq 2,$$

where v_k, $k \geq 2$, are defined by (2.34), $c_1, c_2 \notin \{0, -1, -2, \ldots\}$, μ is a positive number, and $0 < \nu < 1$. Then:

(A) The branched continued fraction (2.33) converges uniformly on every compact subset of the domain (3.47) to a function $f(\mathbf{z})$ holomorphic in (3.47), where

$$\tau = \max_{k \geq 2}\{|v_k|\}.$$

(B) The function $f(\mathbf{z})$ is an analytic continuation of the function (2.24) *in the domain* (3.47).

Now, setting $a=0$ and replacing c_1 by c_1-1 in Theorem 3.8 or setting $a=0$ and replacing c_2 by c_2-1 in Theorem 3.9, we obtain the following result.

Corollary 3.3. *Let c_1 be complex constants such that*

$$|w_k| + \mathrm{Re}(w_k) \le \mu\nu(1-\nu),\ k \ge 1,$$

where w_k, $k \ge 1$, are defined by (2.37), *$c_1 \notin \{0,-1,-2,\ldots\}$, μ is a positive number, and $0<\nu<1$. Then the branched continued fraction* (2.36) *converges uniformly on every compact subset of the domain* (3.47) *to a function $f(\mathbf{z})$ holomorphic in this domain, and, in addition, the function $f(\mathbf{z})$ is an analytic continuation of the function* (2.35) *in the domain* (3.47), *where*

$$\tau = \max_{k\ge1}\{|w_k|\}.$$

Now, we prove the following theorem.

Theorem 3.10. *Let a, c_1, and c_2 be complex constants such that*

$$|h_k| + \mathrm{Re}(h_k) \le \mu\nu(1-\nu),\ k \ge 1, \tag{3.53}$$

where h_k, $k \ge 1$, are defined in (2.39) *and* (2.40), *$c_1, c_2 \notin \{0,-1,-2,\ldots\}$, μ is a positive number, and $0<\nu<1$. Then:*

(A) The branched continued fraction (2.38) *converges uniformly on every compact subset of the domain*

$$\mathcal{H}_{\mu,\nu,\upsilon}^{\kappa,\tau,\upsilon} = \mathcal{H}_{\mu,\nu,\upsilon} \cup \mathcal{H}^{\kappa,\tau,\upsilon} \tag{3.54}$$

to a function $f(\mathbf{z})$ holomorphic in $\mathcal{H}_{\mu,\nu,\upsilon}^{\kappa,\tau,\upsilon}$, where

$$\mathcal{H}_{\mu,\nu,\upsilon} = \left\{\mathbf{z}\in\mathbb{C}^2:\ |z_1| < \frac{1+\cos(\arg(z_1))}{2\mu},\right.$$
$$\left.\mathrm{Re}(z_2 \mathrm{e}^{-(\mathrm{i}/2)\arg(z_1)}) > -\frac{\nu}{2\upsilon}\cos\left(\frac{\arg(z_1)}{2}\right)\right\} \tag{3.55}$$

and

$$\mathcal{H}^{\kappa,\tau,\upsilon} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{\kappa(1-\kappa)}{2\tau}, \ |z_2| < \frac{1-\kappa}{2\upsilon} \right\}, \tag{3.56}$$

with

$$\tau = \sup_{k\geq 1}\{|h_k|\}, \ \upsilon = \max\{d_1, 1\}, \ 0 < \kappa < 1, \tag{3.57}$$

where d_1 *is defined in* (2.39).

(B) The function $f(\mathbf{z})$ *is an analytic continuation of function* (2.25) *in the domain* (3.54).

Proof. We prove Theorem 3.10(A). Since by (3.56) and (3.57) for arbitrary $k \geq 1$

$$|h_k z_1| = |h_k||z_1| < \tau\frac{\kappa(1-\kappa)}{2\tau} = \frac{\kappa(1-\kappa)}{2}$$

and

$$|d_k z_2| = |d_k||z_2| < \upsilon\frac{1-\kappa}{2\upsilon} = \frac{1-\kappa}{2},$$

then from Theorem 3.1(A), where $g_{0,k} = \kappa$, $k \geq 1$, it follows that the branched continued fraction (2.38) converges for $\mathbf{z} \in \mathcal{H}^{\kappa,\tau,\upsilon}$. Moreover, from Theorem 3.1(B) implies that the approximants of the branched continued fraction (2.38) all lie in the region

$$\mathcal{O} = \{w \in \mathbb{C} : \ |w-1| \leq 1\}$$

if $\mathbf{z} \in \mathrm{H}^{\kappa,\tau,\upsilon}$. Thus, by Theorem 1.4, the convergence of the branched continued fraction (2.38) is uniform on every compact subset of (3.56).

Next, we prove the convergence of the branched continued fraction (2.38) in the domain (3.55). Let n be an arbitrary natural number, $\arg(z_1) = \varphi$, and $\mathbf{z}$ be an arbitrary fixed point in the domain (3.55). We prove that

$$\mathrm{Re}(W_k^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}) > (1-\nu)\cos(\varphi/2) \geq c > 0, \ 1 \leq k \leq n, \ n \geq 1, \tag{3.58}$$

where $W_k^{(n)}(\mathbf{z})$, $1 \leq k \leq n$, $n \geq 1$, are defined by (3.25) and (3.26).

From the above assumption it follows that for an arbitrary neighborhood of the point $\mathbf{z}$, $\mathbf{z} \in \mathcal{H}_{\mu,\nu,\upsilon}$, there exists ε, $0 < \varepsilon \le \pi/2$, such that $|\varphi/2| \le \pi/2-\varepsilon$ and, thus,

$$(1-q)\cos(\varphi/2) \ge (1-q)\cos(\varphi/2-\varepsilon) = (1-q)\sin(\varepsilon) = c > 0.$$

If $k = n$, we have

$$\mathrm{Re}(W_n^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}) = \cos(\varphi/2) > (1-\nu)\cos(\varphi/2).$$

Let the first inequality of (3.58) holds for $k = r+1 \le n$. Then, for $k = r$ from (3.25) we obtain

$$W_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2} = \mathrm{e}^{-\mathrm{i}\varphi/2} + d_{r+1}z_2\mathrm{e}^{-\mathrm{i}\varphi/2} + \frac{h_{r+1}z_1\mathrm{e}^{-\mathrm{i}\varphi}}{W_{s+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}}.$$

Using (3.53), (3.55), and Lemma 1.1, we obtain

$$\begin{gathered}\mathrm{Re}(W_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}) = \\ = \cos(\varphi/2) + d_{r+1}\,\mathrm{Re}(z_2\mathrm{e}^{-\mathrm{i}\varphi/2}) + \mathrm{Re}\left(\frac{h_{r+1}|z_1|}{W_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}}\right) > \\ > \cos(\varphi/2) - d_{r+1}\frac{\nu\cos(\varphi/2)}{2\upsilon} - \frac{|h_{r+1}| - \mathrm{Re}(h_{r+1})}{2\,\mathrm{Re}(W_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})}|z_1| > \\ > \cos(\varphi/2) - \frac{\nu\cos(\varphi/2)}{2} - \frac{\mu\nu(1-\nu)}{2(1-\nu)\cos(\varphi/2)}\frac{1+\cos(\varphi)}{2\mu} = \\ = (1-\nu)\cos(\varphi/2).\end{gathered}$$

From the inequalities (3.58) it follows that $W_1^{(n)}(\mathbf{z}) \ne 0$, $n \ge 1$, $\mathbf{z} \in \mathcal{H}_{\mu,\nu,\upsilon}$. This means that the approximants of the branched continued continued (2.38) form a sequence of holomorphic functions in the domain (3.55).

Let $\mathcal{K}$ be an arbitrary compact subset of the domain (3.55), and $f_n(\mathbf{z})$, $n \ge 1$, be approximants of the branched continued fraction (2.38). Then there exists

$$\mathcal{O}_\eta = \{\mathbf{z} \in \mathbb{C}^2 :\ |z_k| < \eta,\ k = 1, 2\},\ \eta > 0,$$

that satisfy $\mathcal{K} \subset \mathcal{O}_\eta$. Then, for arbitrary $n \ge 1$ and for arbitrary

$$\mathbf{z} \in \mathcal{H}_{\mu,\nu,\upsilon} \cap \mathcal{O}_\eta,$$

we obtain

$$|f_n(\mathbf{z})| \le 1 + \frac{|d_0||z_2||\mathrm{e}^{-\mathrm{i}\varphi/2}|}{|W_1^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}|} < 1 + \frac{|d_0|\eta}{\mathrm{Re}(W_1^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})} <$$
$$< 1 + \frac{|d_0|\eta}{(1-\nu)\cos(\varphi/2)} = C(\mathcal{K}).$$

This proves that the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on every compact subset of the domain $\mathcal{H}_{\mu,\nu,\upsilon}$.

Let

$$\mathcal{L}_\chi = \{\mathbf{z} \in \mathbb{R}^2 : \; 0 < z_k < \chi, \; k = 1, 2\},$$

where

$$0 < \chi < \min\left\{\frac{\kappa(1-\kappa)}{4\tau}, \frac{1-\kappa}{4\upsilon}, \frac{1}{2\mu}, \frac{\nu}{4\upsilon}\right\}.$$

Then it is clear that the domain $\mathcal{L}_\chi$ contained in $\mathcal{H}_{\mu,\nu,\upsilon}$, in particular $\mathcal{L}_{\chi/2} \subset \mathcal{H}_{\mu,\nu,\upsilon}$. From the above proof it follows that branched continued fraction (2.38) converges in the domain $\mathcal{L}_\chi$, $\mathcal{L}_\chi \subset \mathcal{H}_{\mu,\nu,\upsilon}$. By Theorem 1.4, the convergence of the branched continue fraction (2.38) is uniform on compact subsets of the $\mathcal{H}_{\mu,\nu,\upsilon}$, and hence (3.54), which proves Theorem 3.10(A).

Finally, from Theorem 3.4(B) it follows that the branched continued fraction (2.38) corresponds at $\mathbf{z} = \mathbf{0}$ to the function (2.25). Therefore, according to Theorem 1.3 and Theorem 3.10(A), Theorem 3.10(B) follows. ■

Next, we prove the following theorem.

Theorem 3.11. *Let a, c_1, and c_2 be complex constants such that*

$$|h_k| + 2|d_k| + \mathrm{Re}(h_k + 2d_k) \le \frac{\varrho}{2}, \; k \ge 1, \tag{3.59}$$

where h_k, d_k, $k \ge 1$, are defined in (2.39) *and* (2.40)*, $c_1, c_2 \notin \{0, -1, -2, \ldots\}$, ϱ is a positive number. Then:*

(A) The branched continued fraction (2.38) *converges uniformly on every compact subset of the domain*

$$\mathcal{H}_\varrho^{\kappa,\tau,\upsilon} = \mathcal{H}_\varrho \cup \mathcal{H}^{\kappa,\tau,\upsilon} \tag{3.60}$$

to a function $f(\mathbf{z})$ *holomorphic in* $\mathcal{H}_{\varrho}^{\kappa,\tau,\upsilon}$, *where*

$$\mathcal{H}_{\varrho}=\left\{\mathbf{z}\in\mathbb{C}^2:\ |z_k|<\frac{1+\cos(\arg(z_k))}{\varrho},\right.$$
$$\left.\arg(z_1)=\arg(z_2),\ k=1,2\right\} \tag{3.61}$$

and $\mathcal{H}^{\kappa,\tau,\upsilon}$ *is defined by* (3.56) *with* (3.57).

(B) The function $f(\mathbf{z})$ *is an analytic continuation of function* (2.25) *in the domain* (3.60).

Proof. By Theorem 3.10(A), the statement of Theorem 3.11(A) is valid in the domain (3.56). We will show the validity of Theorem 3.11(A) in the domain (3.61).

We set $\arg(z_1)=\varphi$. Let n be an arbitrary natural number, $\mathbf{z}$ be an arbitrary fixed point in (3.61). In the neighborhood of $\mathbf{z}$, by induction on k, we prove the following inequalities

$$\mathrm{Re}(W_k^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})>\frac{\cos(\varphi/2)}{2}\ge c>0,\ 1\le k\le n,\ n\ge 1, \tag{3.62}$$

where $W_k^{(n)}(\mathbf{z})$, $1\le k\le n$, $n\ge 1$, are defined by (3.25) and (3.26).

Since $\mathbf{z}$ is an arbitrary fixed point from domain (3.61), then for its arbitrary neighborhood there exists ε, $0<\varepsilon\le\pi/2$, such that $|\varphi/2|\le\pi/2-\varepsilon$ and, therefore,

$$\frac{\cos(\varphi/2)}{2}\ge\frac{\cos(\pi/2-\varepsilon)}{2}=\frac{\sin(\varepsilon)}{2}=c>0.$$

Let us show the validity of the first inequality in (3.61). At $k=n$, this inequality is obvious. Assuming that the first inequality in (3.62) are true if $k=r+1\le n$, we show it for $k=r$. From (3.26) we obtain

$$W_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}=\mathrm{e}^{-\mathrm{i}\varphi/2}+\frac{d_{r+1}z_2\mathrm{e}^{-\mathrm{i}\varphi}}{\mathrm{e}^{-\mathrm{i}\varphi/2}}+\frac{h_{r+1}z_1\mathrm{e}^{-\mathrm{i}\varphi}}{W_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}}. \tag{3.63}$$

Using Lemma 1.1, (3.59), (3.61), from (3.63) we have

$$\mathrm{Re}(W_r^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2}) =$$
$$= \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\varphi/2}) - \mathrm{Re}\left(\frac{d_{r+1}z_2\mathrm{e}^{-\mathrm{i}\varphi}}{\mathrm{e}^{-\mathrm{i}\varphi/2}}\right) - \mathrm{Re}\left(\frac{h_{r+1}z_1\mathrm{e}^{-\mathrm{i}\varphi/2}}{W_{r+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})}\right) \geq$$
$$\geq \cos(\varphi/2) - \frac{|d_{r+1}| - \mathrm{Re}(d_{r+1})}{2\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\varphi/2})}|z_2| - \frac{|h_{s+1}| - \mathrm{Re}(h_{r+1})}{2\,\mathrm{Re}(W_{s+1}^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})}|z_1|$$
$$> \cos(\varphi/2) - [|d_{r+1}| - \mathrm{Re}(d_{r+1}) + 2(|h_{s+1}| - \mathrm{Re}(h_{r+1}))]\frac{1+\cos(\varphi)}{2\varrho\cos(\varphi/2)}$$
$$> \cos(\varphi/2) - \frac{\cos(\varphi/2)}{2} = \frac{\cos(\varphi/2)}{2}.$$

Thus, $W_1^{(n)}(\mathbf{z}) \neq 0$ for all $n \geq 1$ and $\mathbf{z} \in \mathcal{H}_\varrho$, i.e., that each approximant the branched continued fraction (2.38) is a holomorphic function in (3.61).

Let $\mathcal{K}$ be an arbitrary compact subset of (3.61), and $f_n(\mathbf{z})$, $n \geq 1$, be approximants of the branched continued fraction (2.38). Then there exists an open bi-disk

$$\mathcal{O}_\eta = \{\mathbf{z} \in \mathbb{C}^2 : \ |z_k| < \eta,\ k = 1, 2\}$$

of radius η, $\eta > 0$, such that $\mathcal{K} \subset \mathcal{O}_\eta$. Moreover, for any $n \geq 1$ and $\mathbf{z} \in \mathcal{H}_\varrho \cap \mathcal{O}_\eta$ we obtain

$$|f_n(\mathbf{z})| \leq 1 + \frac{|d_0|\eta}{\mathrm{Re}(W_1^{(n)}(\mathbf{z})\mathrm{e}^{-\mathrm{i}\varphi/2})} <$$
$$< 1 + \frac{2|d_0|\eta}{\cos(\varphi/2)} = C(\mathcal{K}),$$

i.e., the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on every compact subset of the domain $\mathcal{H}_\varrho$.

Since

$$d_1 = \frac{a}{c_2+1} - 1,\ d_k = -1,\ k \geq 2,\ \text{and}\ \lim_{k\to+\infty} h_k = -1,$$

then exists a constant $\zeta > 0$ such that

$$|d_k| \leq \zeta \text{ and } |h_k| \leq \zeta,\ k \geq 1. \tag{3.64}$$

It is clear that for every χ such that

$$0 < \chi < \min\left\{\frac{1}{4\zeta}, \frac{1}{8\zeta}, \frac{2}{\varrho}\right\}$$

the domain

$$\mathcal{L}_\chi = \{\mathbf{z} \in \mathbb{R}^2 : \ 0 < z_k < \chi, \ k = 1, 2\}$$

contained in $\mathcal{H}_\varrho$, in particular $\mathcal{L}_{\chi/2} \subset \mathcal{H}_\varrho$.

Using (3.64), for any $k \geq 1$ and $\mathbf{z} \in \mathcal{L}_\chi$, $\mathcal{L}_\chi \subset \mathcal{H}_\varrho$, it is clear that

$$|d_k z_2| < \frac{1}{4}, \ |h_k z_1| < \frac{1}{8}, \ k \geq 1,$$

i.e. the elements of the branched continued fraction (2.38) satisfy Theorem 3.1, with

$$z_{1,0} = 0, \ g_{0,k} = \frac{1}{2}, \ k \geq 1.$$

It follows from this theorem that the branched continued fraction (2.38) converges in the domain $\mathcal{L}_\chi$, $\mathcal{L}_\chi \subset \mathcal{H}_\varrho$, Therefore, by Theorem 1.4 the convergence of the branched continued fraction (2.38) is uniform on compact subsets of $\mathcal{H}_\varrho$, and hence (3.56), which proves Theorem 3.11(A).

Theorem 3.11(B) follows from Theorem 1.3, Theorem 3.4(B), and Theorem 3.11(A). ■

Finally, we consider the following example.

Example 3.4. *Using (h) in [89, Subsection 2.10] and the Mellin-Barnes integral [79, Formula (15.3.2)], by Theorem 3.10 we have*

$$\frac{H_4(1/2, 2; 1/2, 1; \mathbf{z})}{H_4(1/2, 3; 1/2, 2; \mathbf{z})} =$$

$$= \frac{2\displaystyle\int_{-i\infty}^{+i\infty} \frac{\Gamma(\xi + 1/2)\Gamma(\xi + 2)\Gamma(-\xi)}{\Gamma(\xi + 1)} h(\xi; \mathbf{z}) d\xi}{\displaystyle\int_{-i\infty}^{+i\infty} \frac{\Gamma(\xi + 1/2)\Gamma(\xi + 3)\Gamma(-\xi)}{\Gamma(\xi + 2)} h(\xi; \mathbf{z}) d\xi} =$$

$$= 1 - \cfrac{\frac{1}{4}z_2}{1 + \frac{3}{4}z_2 + \cfrac{6z_1}{1 + z_2 - \cfrac{\frac{5}{3}z_1}{1 + z_2 - \cfrac{\frac{7}{15}z_1}{1 + \ddots}}}}, \tag{3.65}$$

where

$$h(\xi;\mathbf{z}) = (1-2\sqrt{z_1})^{-\mathrm{i}/2}\left(\frac{-z_2}{1-2\sqrt{z_1}}\right)^{-\xi} + (1+2\sqrt{z_1})^{-\mathrm{i}/2}\left(\frac{-z_2}{1+2\sqrt{z_1}}\right)^{-\xi}.$$

Here the branched continued fraction converges and represents a single-valued branch of the analytic function on the left side of (3.65) *in the domain* $\mathcal{H}_{\mu,\nu,\upsilon}^{\kappa,\tau,\upsilon} \cap \mathcal{H}$*, where* $\mathcal{H}_{\mu,\nu,\upsilon}^{\kappa,\tau,\upsilon}$ *is defined by (3.54) with*

$$\mu\nu(1-\nu) \geq \frac{10}{3},\ \upsilon = 1,\ \tau = 6,$$

and

$$\mathcal{H} = \left\{\mathbf{z} \in \mathbb{C}^2 :\ \left|\arg\left(\frac{-z_2}{1 \pm 2\sqrt{z_1}}\right)\right| < \pi,\ \mathrm{Re}(1 \pm 2\sqrt{z_1}) > 0\right\} \tag{3.66}$$

(the principal branch of the square root is assumed). In addition, by Theorem 3.11 the branched continued fraction (3.65) converges and represents a single-valued branch of the analytic function on the left side of (3.65) in the domain $\mathcal{H}_{\varrho}^{\kappa,\tau,\upsilon} \cap \mathcal{H}$*, where* $\mathcal{H}_{\varrho}^{\kappa,\tau,\upsilon}$ *is defined by (3.60) with*

$$\varrho \geq \frac{40}{3},\ \upsilon = 1,\ \tau = 6,$$

and $\mathcal{H}$ *is defined by (3.66), and, thus, in the domain*

$$\left(\mathcal{H}_{\mu,\nu,\upsilon}^{\kappa,\tau,\upsilon} \cup \mathcal{H}_{\varrho}^{\kappa,\tau,\upsilon}\right) \cap \mathcal{H}$$

with

$$\mu\nu(1-\nu) \geq \frac{10}{3},\ \varrho \geq \frac{40}{3},\ \upsilon = 1,\ \tau = 6.$$

This chapter establishes the domains of analytic continuation of the Horn hypergeometric functions H_4 and their ratios in special cases that have expansions into branched continued fractions. These domains are obtained as domains of convergence of the above-mentioned expansions. Examples of analytical functions represented by branched continued fractions are also given here. The obtained results can be used to approximate analytic functions represented by the Horn hypergeometric series H_4 and their ratios in special cases.

The results presented in this chapter were published in [57,61,63–65,67,68].

CHAPTER 4

CONVERGENCE RATE AND NUMERICAL STABILITY

The chapter investigates the convergence rate and the numerical stability of branched continued fraction expansions of Horn hypergeometric series H_4 and their ratios in special cases. These problems have considerable importance for the computation of functions using branched continued fraction representations. In Section 4.1 we find truncation error bounds of the approximants of the above-mention expansions, and in Section 4.2, we find estimates of the relative roundoff errors in computing these approximants by the backward recurrence algorithm.

4.1. Truncation error bounds

The following theorem holds.

Theorem 4.1. *Let a and c_1 be real constants satisfying the inequalities* (3.9), *where u_k, $k \geq 1$, are defined by* (2.27), *τ is a positive number. Then:*

(A) The branched continued fraction (2.26) *converges to a finite value $f(\mathbf{z})$ for each $\mathbf{z} \in \mathcal{R}_\kappa$, where*

$$\mathcal{R}_\kappa = \left\{\mathbf{z} \in \mathbb{R}^2 : \ z_1 \leq 0, \ z_2 \leq \kappa\right\}, \ 0 < \kappa < 1. \tag{4.1}$$

(B) The convergence is uniform on every compact subset of $\operatorname{Int}(\mathcal{R}_\kappa)$, *and the function $f(\mathbf{z})$ is analytic on* $\operatorname{Int}(\mathcal{R}_\kappa)$.

(C) If $f_n(\mathbf{z})$ denotes the nth approximant of the branched continued fraction (2.26), *then for each $\mathbf{z} \in \mathcal{R}_\kappa$*

$$|f(\mathbf{z}) - f_n(\mathbf{z})| \leq \frac{|u_1|(|z_2|(1-z_2) + \tau|z_1|)\tau^{n-1}|z_1|^n}{(1-z_2)^3(1-z_2+\tau|z_1|)((1-z_2)^2+\tau|z_1|)^{n-2}}, \ n \geq 2.$$

(D) The function $f(\mathbf{z})$ is an analytic continuation of the function (2.23) *in the domain* $\operatorname{Int}(\mathcal{R}_\kappa)$.

Proof. First, we will prove Theorem 4.1(A). To this, for $n \geq 1$ and $k \geq 1$ we estimate

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})|.$$

Let $\mathbf{z}$ be an arbitrary fixed point in the region (4.1). It is obvious that under conditions (4.1) the coefficients (2.27) are positive. Thus, by inequalities in (4.1) from (3.11) and (3.13) for $1 \leq k \leq n-1$ and $n \geq 2$ we have

$$F_k^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{u_{k+1} z_1}{F_{k+1}^{(n)}(\mathbf{z})} \geq 1 - z_2 \geq 1 - \kappa > 0. \tag{4.2}$$

Condition (4.2) allows us to use the formula (1.15), namely for any $n \geq 2$ and $k \geq 1$

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = -z_1^n \left(z_2 + \frac{u_{n+1} z_1}{F_{n+1}^{(n+k)}(\mathbf{z})} \right) \prod_{r=1}^{n} \frac{u_r}{F_r^{(n+k)}(\mathbf{z}) F_r^{(n)}(\mathbf{z})}.$$

For convenience, we write this formula as

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = -\frac{u_1 z_1^n}{F_1^{(q)}(\mathbf{z}) F_n^{(n+k)}(\mathbf{z})} \left(z_2 + \frac{u_{n+1} z_1}{F_{n+1}^{(n+k)}(\mathbf{z})} \right) \times$$
$$\times \prod_{r=1}^{[n/2]} \frac{u_{2r}}{F_{2r-1}^{(p)}(\mathbf{z}) F_{2r}^{(p)}(\mathbf{z})} \prod_{r=1}^{[(n-1)/2]} \frac{u_{2r+1}}{F_{2r}^{(q)}(\mathbf{z}) F_{2r+1}^{(q)}(\mathbf{z})}, \tag{4.3}$$

where [.] denotes integer part, $q = n+k$, $p = n$, if $n = 2s$, and $q = n$, $p = n+k$, if $n = 2s-1$, $s \geq 1$.

By the inequalities (4.1) and (4.2) for any $m \geq 2$ and $k \geq 2$ we obtain

$$\frac{u_1 z_1}{F_1^{(m)}(\mathbf{z})} \leq \frac{|u_1||z_1|}{F_1^{(m)}(\mathbf{z})} \leq \frac{|u_1||z_1|}{1 - z_2}$$

and

$$\frac{1}{F_m^{(m+k)}(\mathbf{z})} \left(z_2 + \frac{u_{m+1} z_1}{F_{m+1}^{(m+k)}(\mathbf{z})} \right) \leq \frac{1}{F_m^{(m+k)}(\mathbf{z})} \left(|z_2| + \frac{u_{m+1}|z_1|}{F_{m+1}^{(m+k)}(\mathbf{z})} \right) \leq$$
$$\leq \frac{1}{1 - z_2} \left(|z_2| + \frac{\tau |z_1|}{1 - z_2} \right).$$

Moreover, by the relations (3.13) for any $1 \le k \le m-1$ and $m \ge 2$ we have

$$\frac{u_{k+1}z_1}{F_k^{(m+1)}(\mathbf{z})F_{k+1}^{(m+1)}(\mathbf{z})} = \frac{\dfrac{u_{k+1}z_1}{F_{k+1}^{(m+1)}(\mathbf{z})}}{1 - z_2 - \dfrac{u_{k+1}z_1}{F_{k+1}^{(m+1)}(\mathbf{z})}} \le$$

$$\le \frac{\dfrac{u_{k+1}|z_1|}{F_{k+1}^{(m+1)}(\mathbf{z})}}{1 - z_2 + \dfrac{u_{k+1}|z_1|}{F_{k+1}^{(m+1)}(\mathbf{z})}} \le \frac{\tau|z_1|}{(1-z_2)^2 + \tau|z_1|}.$$

Finally, by the relations (3.11), (3.13) and the inequalities (3.9), (4.2) for any $m \ge 2$ we get

$$\frac{u_{m+1}z_1}{F_m^{(m+1)}(\mathbf{z})F_{m+1}^{(m+1)}(\mathbf{z})} = \frac{u_{m+1}z_1}{\left(1 - z_2 - \dfrac{u_{m+1}z_1}{F_{m+1}^{(m+1)}}\right)F_{m+1}^{(m+1)}} =$$

$$= \frac{u_{m+1}z_1}{1 - z_2 - u_{m+1}z_1} \le \frac{\tau|z_1|}{1 - z_2 + \tau|z_1|}.$$

Thus, from (4.3) for $n \ge 2$ and $k \ge 2$ we obtain

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| \le \frac{|u_1||z_1|^n}{F_1^{(q)}(\mathbf{z})F_n^{(n+k)}(\mathbf{z})}\left(|z_2| + \frac{u_{n+1}|z_1|}{F_{n+1}^{(n+k)}(\mathbf{z})}\right)\times$$

$$\times \prod_{r=1}^{[n/2]} \frac{u_{2r}}{F_{2r-1}^{(p)}(\mathbf{z})F_{2r}^{(p)}(\mathbf{z})} \prod_{r=1}^{[(n-1)/2]} \frac{u_{2r+1}}{F_{2r}^{(q)}(\mathbf{z})F_{2r+1}^{(q)}(\mathbf{z})} \le$$

$$\le \frac{|u_1|(|z_2|(1-z_2) + \tau|z_1|)\tau^{n-1}|z_1|^n}{(1-z_2)^3(1-z_2+\tau|z_1|)((1-z_2)^2+\tau|z_1|)^{n-2}}. \tag{4.4}$$

Since for an arbitrary fixed $\mathbf{z} \in \mathcal{R}_\kappa$

$$\frac{|u_1|(|z_2|(1-z_2) + \tau|z_1|)\tau^{n-1}|z_1|^n}{(1-z_2)^3(1-z_2+\tau|z_1|)((1-z_2)^2+\tau|z_1|)^{n-2}} \to 0$$

as $n \to +\infty$, then from the arbitrariness of k it follows Theorem 4.1(A).

Now, we will prove Theorem 4.1(B). Let $\mathcal{K}$ be an arbitrary compact subset of the domain $\mathrm{Int}(\mathcal{R}_\kappa)$. Then there exists an open bi-disk

$$\mathcal{O}_\eta = \{\mathbf{z} \in \mathbb{C}^2 : \ |z_k| < \eta, \ k = 1, 2\}$$

of radius η, $\eta > 0$, such that $\mathcal{K} \subset \mathcal{O}_\eta$. Then, for $n \geq 2$ and $k \geq 2$ we have

$$\begin{aligned} &|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| \leq \\ &\leq \frac{|u_1|(|z_2|(1-z_2)+\tau|z_1|)\tau^{n-1}|z_1|^n}{(1-z_2)^3(1-z_2+\tau|z_1|)((1-z_2)^2+\tau|z_1|)^{n-2}} < \\ &< \frac{|u_1|(\eta(1-\kappa)+\tau\eta)\tau^{n-1}\eta^n}{(1-\kappa)^3(1-\kappa+\tau\eta)((1-\kappa)^2+\tau\eta)^{n-2}} \\ &= \frac{|u_1|(1-\kappa+\tau)\tau^{n-1}\eta^{n+1}}{(1-\kappa)^3(1-\kappa+\tau\eta)((1-\kappa)^2+\tau\eta)^{n-2}} \end{aligned}$$

for all $\mathbf{z} \in \mathcal{K}$. Moreover, if q and p are arbitrary integer numbers such that $q \geq 2$ and $p \geq n \geq 2$, then, for all $\mathbf{z} \in \mathcal{K}$,

$$|f_{p+q}(\mathbf{z}) - f_p(\mathbf{z})| \leq |f_{p+q}(\mathbf{z}) - f_n(\mathbf{z})| + |f_p(\mathbf{z}) - f_n(\mathbf{z})|.$$

Since

$$\frac{|u_1|(1-\kappa+\tau)\tau^{n-1}\eta^{n+1}}{(1-\kappa)^3(1-\kappa+\tau\eta)((1-\kappa)^2+\tau\eta)^{n-2}} \to 0$$

as $n \to +\infty$, this proves Theorem 4.1(B).

Theorem 4.1(C) follows from (4.4) if we pass to the limit as $k \to +\infty$.

Next, we will prove Theorem 4.1(D). It is clear that

$$\frac{H_4(a, c_2; c_1, c_2; \mathbf{0})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{0})} = 1.$$

Then, there exists $0 < \varepsilon < 1$ such that function (2.23) is analytic in domain

$$\mathcal{D}_{r,s,\varepsilon} = \left\{\mathbf{z} \in \mathbb{R}^2 : \ -r\varepsilon < z_1 < 0, \ -s\varepsilon < z_2 < 0\right\} \tag{4.5}$$

and

$$\mathcal{D}_{r,s,\varepsilon} \subset (\mathcal{D}_{r,s} \cap \operatorname{Int}(\mathcal{R}_\kappa)),$$

where

$$\mathcal{D}_{r,s} = \left\{\mathbf{z} \in \mathbb{C}^2 : \ |z_1| < r, \ |z_2| < s\right\}, \tag{4.6}$$

r and s are positive numbers such that $4r = (s-1)^2$ herewith $s \neq 1$, $\mathcal{R}_\kappa$ is defined by (4.1). In particular,

$$\mathcal{D}_{r,s,1/2} \subset (\mathcal{D}_{r,s} \cap \operatorname{Int}(\mathcal{D}_\eta)).$$

Let $\mathbf{z}$ be an arbitrary fixed point in $\mathcal{D}_{r,s,\varepsilon}$. It is obvious that the elements of the branched continued fraction (2.26) are positive. Then the approximants of this branched continued fraction have the property of fork, namely

$$f_{2n}(\mathbf{z}) < f_{2n+2}(\mathbf{z}) < f_{2n+1}(\mathbf{z}) < f_{2n-1}(\mathbf{z}),\ n \geq 1,$$

and, therefore, the sequences of even and odd approximants of branched continued fraction (2.26) converge to a finite value $f(\mathbf{z})$.

Now, we will consider

$$\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})} - f_n(\mathbf{z}),\ n \geq 1,$$

where (see, Theorem 2.7)

$$\frac{H_4(a, b; c, b; \mathbf{z})}{H_4(a+1, b; c+1, b; \mathbf{z})} =$$

$$= 1 - z_2 - \cfrac{u_1 z_1}{1 - z_2 - \cfrac{u_2 z_1}{1 - \ddots - z_2 - \cfrac{u_{n+1} z_1}{E_{n+1}^{(n+1)}(\mathbf{z})}}},$$

and, for $n \geq 1$,

$$E_{n+1}^{(n+1)}(\mathbf{z}) = \frac{H_4(a+n+1, c_2; c_1+n+1, c_2; \mathbf{z})}{H_4(a+n+2, c_2; c_1+n+2, c_2; \mathbf{z})}.$$

We set

$$E_k^{(n+1)}(\mathbf{z}) = 1 - z_2 - \cfrac{u_{k+1} z_1}{1 - z_2 - \cfrac{u_{k+2} z_1}{1 - \ddots - z_2 - \cfrac{u_{n+1} z_1}{E_{n+1}^{(n+1)}(\mathbf{z})}}},$$

where $1 \leq k \leq n$, $n \geq 1$. Then the following relation

$$E_k^{(n+1)}(\mathbf{z}) = 1 - z_2 - \frac{u_{k+1} z_1}{E_{k+1}^{(n+1)}(\mathbf{z})},\ 1 \leq k \leq n,\ n \geq 1,$$

is valid.

It is easy to see that $F_k^{(n)}(\mathbf{z}) \neq 0$ and $E_k^{(n)}(\mathbf{z}) \neq 0$ for all indices and $\mathbf{z} \in \mathcal{D}_{r,s,\varepsilon}$. Using the formula (1.15), for $n \geq 1$ we obtain

$$\frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})} - f_n(\mathbf{z}) =$$
$$= -z_1^n \left(z_2 + \frac{u_{n+1} z_1}{E_{n+1}^{(n+1)}(\mathbf{z})} \right) \prod_{r=1}^{n} \frac{u_r}{E_r^{(n+1)}(\mathbf{z}) F_r^{(n)}(\mathbf{z})}.$$

Thus, for all $\mathbf{z} \in \mathcal{D}_{r,s,\varepsilon}$ we obtain

$$f_{2n}(\mathbf{z}) < \frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})} < f_{2n-1}(\mathbf{z}), \ n \geq 1.$$

Next, since the fork property of approximants of branched continued fraction (2.24) implies that for all $\mathbf{z} \in \mathcal{D}_{r,s,\varepsilon}$

$$\lim_{n \to +\infty} f_{2n}(\mathbf{z}) = \lim_{n \to +\infty} f_{2n-1}(\mathbf{z}) = f(\mathbf{z}),$$

then also for all $\mathbf{z} \in \mathcal{D}_{r,s,\varepsilon}$,

$$f(\mathbf{z}) = \frac{H_4(a, c_2; c_1, c_2; \mathbf{z})}{H_4(a+1, c_2; c_1+1, c_2; \mathbf{z})}.$$

Finally, by Theorem 1.3 and Theorem 4.1(B), Theorem 4.1(D) follows. ■

By Theorems 3.2 and 3.5, Theorem 4.1(B),(D), and Theorem 1.3 we have the following result.

Theorem 4.2. *Let the conditions of Theorem 4.1 be satisfied. Then the branched continued fraction* (2.26) *converges uniformly on every compact subset of the domain*

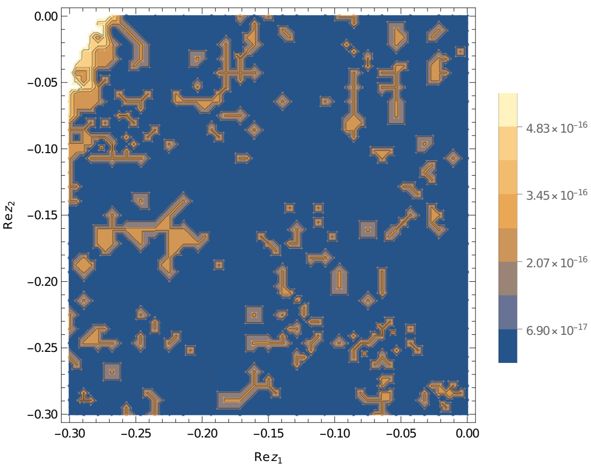
$$\mathcal{D}_\tau \cup \mathcal{P}_\tau \cup \mathrm{Int}(\mathcal{R}_\kappa), \tag{4.7}$$

to a function $f(\mathbf{z})$ holomorphic in this domain, and, in addition, the function $f(\mathbf{z})$ is an analytic continuation of the function (2.23) *in this domain, where $\mathcal{D}_\tau$, $\mathcal{P}_\tau$, and $\mathcal{R}_\kappa$ are defined by* (3.10), (3.34), *and* (4.1), *respectively.*

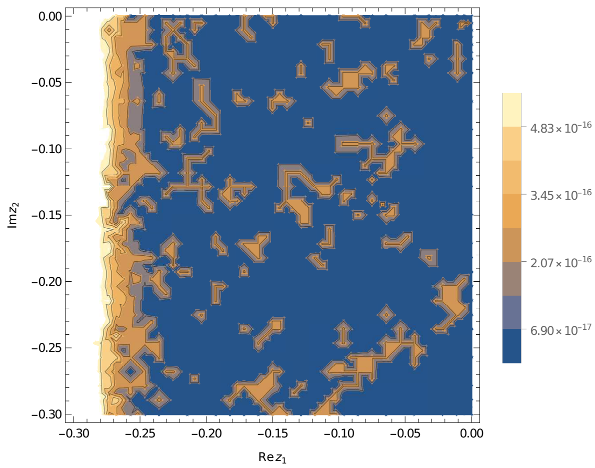
We consider the following example.

Example 4.1. *From Example 3.1 and Theorem 4.2 the branched continued fraction* (3.36) *converges and represents a single-valued branch of the analytic function* (3.37) *in the domain* $\mathcal{P} \cup \mathrm{Int}(\mathcal{R}_\kappa)$, *where* $\mathcal{P}$ *and* $\mathcal{R}_\kappa$ *are defined by* (3.38) *and* (4.1), *respectively.*

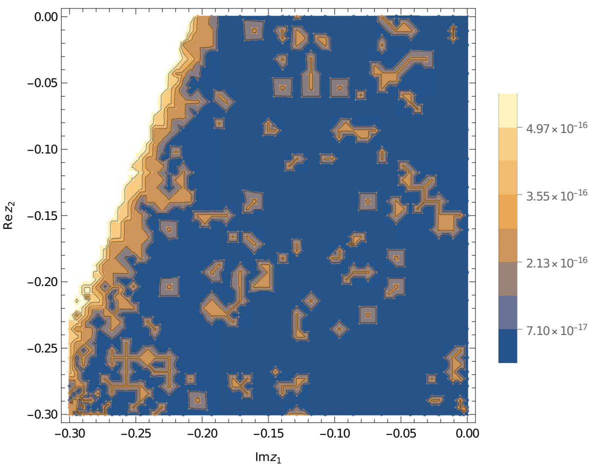
Figure 4.1(a)–(d) shows the plots, where the 20th approximants of branched continued fraction (3.36) *guarantees certain truncation error bounds for function of two variables* (3.37).



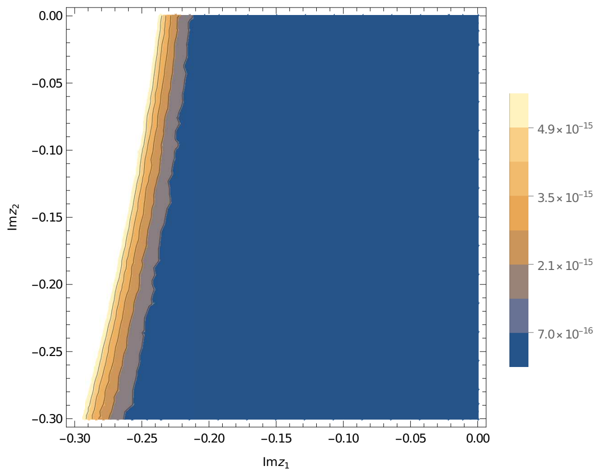
(**a**) In the $(\mathrm{Re}(z_1), \mathrm{Re}(z_2))$ plane.



(**b**) In the $(\mathrm{Re}(z_1), \mathrm{Im}(z_2))$ plane.



(**c**) In the $(\mathrm{Im}(z_1), \mathrm{Re}(z_2))$ plane.



(**d**) In the $(\mathrm{Im}(z_1), \mathrm{Im}(z_2))$ plane.

Figure 4.1. The plots where the approximant $f_{20}(\mathbf{z})$ of branched continued fraction (3.36) guarantees certain truncation error bounds for the function (3.37).

The following result can be proved in much the same way as Theorem 4.1.

Theorem 4.3. *Let* a, c_1, c_2 *be real constants satisfying the inequalities* (3.22), *where* v_k, $k \geq 1$, *are defined by* (2.34), τ *is a positive number. Then:*

(A) The branched continued fraction (2.33) *converges to a finite value* $f(\mathbf{z})$ *for each* $\mathbf{z} \in \mathcal{R}_\kappa$, *where* $\mathcal{R}_\kappa$ *is defined by* (4.1).

(B) The convergence is uniform on every compact subset of $\mathrm{Int}(\mathcal{R}_\kappa)$, *and the function* $f(\mathbf{z})$ *is analytic on* $\mathrm{Int}(\mathcal{R}_\kappa)$.

(C) If $f_n(\mathbf{z})$ *denotes the nth approximant of the branched continued fraction* (2.33), *then for each* $\mathbf{z} \in \mathcal{R}_\kappa$

$$|f(\mathbf{z}) - f_n(\mathbf{z})| \leq$$
$$\leq \frac{|v_1|(|z_2|(1-z_2)+\tau|z_1|)\tau^{n-1}|z_1|^n}{(1-z_2)^3(1-z_2+\tau|z_1|)((1-z_2)^2+\tau|z_1|)^{n-2}}, \ n \geq 2.$$

(D) The function $f(\mathbf{z})$ *is an analytic continuation of the function* (2.24) *in the domain* $\mathrm{Int}(\mathcal{R}_\kappa)$.

By Theorems 3.3 and 3.6, Theorem 4.3(B),(D), and Theorem 1.3, we have the following result.

Theorem 4.4. *Let the conditions of Theorem 4.3 be satisfied. Then the branched continued fraction* (2.33) *converges uniformly on every compact subset of the domain* (4.7) *to a function* $f(\mathbf{z})$ *holomorphic in this domain, and, in addition, the function* $f(\mathbf{z})$ *is an analytic continuation of the function* (2.24) *in this domain, where* $\mathcal{D}_\tau$, $\mathcal{P}_\tau$, *and* $\mathcal{R}_\kappa$ *are defined by* (3.10), (3.34), *and* (4.1), *respectively.*

Setting $a = 0$ and replacing c_1 by $c_1 - 1$ in Theorem 4.1 or setting $a = 0$ and replacing c_2 by $c_2 - 1$ in Theorem 4.3, we have the following corollary.

Corollary 4.1. *Let* c *be real constant satisfying the inequalities* (3.23), *where* w_k, $k \geq 1$, *are defined by* (2.37), τ *is a positive number. Then:*

(A) The branched continued fraction (2.36) *converges to a finite value* $f(\mathbf{z})$ *for each* $\mathbf{z} \in \mathcal{R}_\kappa$, *where* $\mathcal{R}_\kappa$ *is defined by* (4.1).

(B) The convergence is uniform on every compact subset of $\mathrm{Int}(\mathcal{R}_\kappa)$*, and the function* $f(\mathbf{z})$ *is analytic on* $\mathrm{Int}(\mathcal{R}_\kappa)$*.*

(C) If $f_n(\mathbf{z})$ *denotes the nth approximant of the branched continued fraction* (2.36)*, then for each* $\mathbf{z} \in \mathcal{R}_\kappa$

$$|f(\mathbf{z}) - f_n(\mathbf{z})| \leq$$
$$\leq \frac{(|z_2|(1-z_2) + \tau|z_1|)(\tau|z_1|)^{n-1}}{(1-z_2)^5(1-z_2+\tau|z_1|)((1-z_2)^2+\tau|z_1|)^{n-3}}, \ n \geq 3.$$

(D) The function $f(\mathbf{z})$ *is an analytic continuation of the function* (2.35) *in the domain* $\mathrm{Int}(\mathcal{R}_\kappa)$*.*

By Corollaries 3.1 and 3.2, Corollary 4.1(B),(D), and Theorem 1.3, we have the following result.

Corollary 4.2. *Let the conditions of Corollary 4.1 be satisfied. Then the branched continued fraction* (2.36) *converges uniformly on every compact subset of the domain* (4.7) *to a function* $f(\mathbf{z})$ *holomorphic in this domain, and, in addition, the function* $f(\mathbf{z})$ *is an analytic continuation of the function* (2.35) *in this domain, where* $\mathcal{D}_\tau$*,* $\mathcal{P}_\tau$*, and* $\mathcal{R}_\kappa$ *are defined by* (3.10)*,* (3.34)*, and* (4.1)*, respectively.*

We consider the following example.

Example 4.2. *From Example 3.2, Theorem 4.1, and Theorem 1.3 the branched continued fraction* (3.39) *converges and represents a single-valued branch of the analytic function* (3.40) *in the domain* $\mathcal{P} \cup \mathrm{Int}(\mathcal{R}_\kappa)$*, where* $\mathcal{P}$ *and* $\mathcal{R}_\kappa$ *are defined by* (3.38) *and* (4.1)*, respectively.*

If $f_n(\mathbf{z})$ *denotes the nth approximant of the branched continued fraction* (3.39)*, then for every negative real* $\mathbf{z} = \mathbf{z}^0$*, the fork property, namely,*

$$f_{2k-2}(\mathbf{z}^0) < f_{2k}(\mathbf{z}^0) < f_{2k+1}(\mathbf{z}^0) < f_{2k-1}(\mathbf{z}^0), \ k \geq 1,$$

holds (here $f_0(\mathbf{z}^0) = 0$*).*

The numerical illustration of the double power series

$$((1-z_2)^2-4z_1)^{-1/2}=H_4(1,c_1;1,c_2;\mathbf{z})=$$
$$=\sum_{p,q=0}^{\infty}\frac{(1)_{2p+q}}{(1)_p}\frac{z_1^p}{p!}\frac{z_2^q}{q!} \tag{4.8}$$

and the branched continued fraction (3.39) *is given in Table 4.1. Numerical experiments also show that to compute*

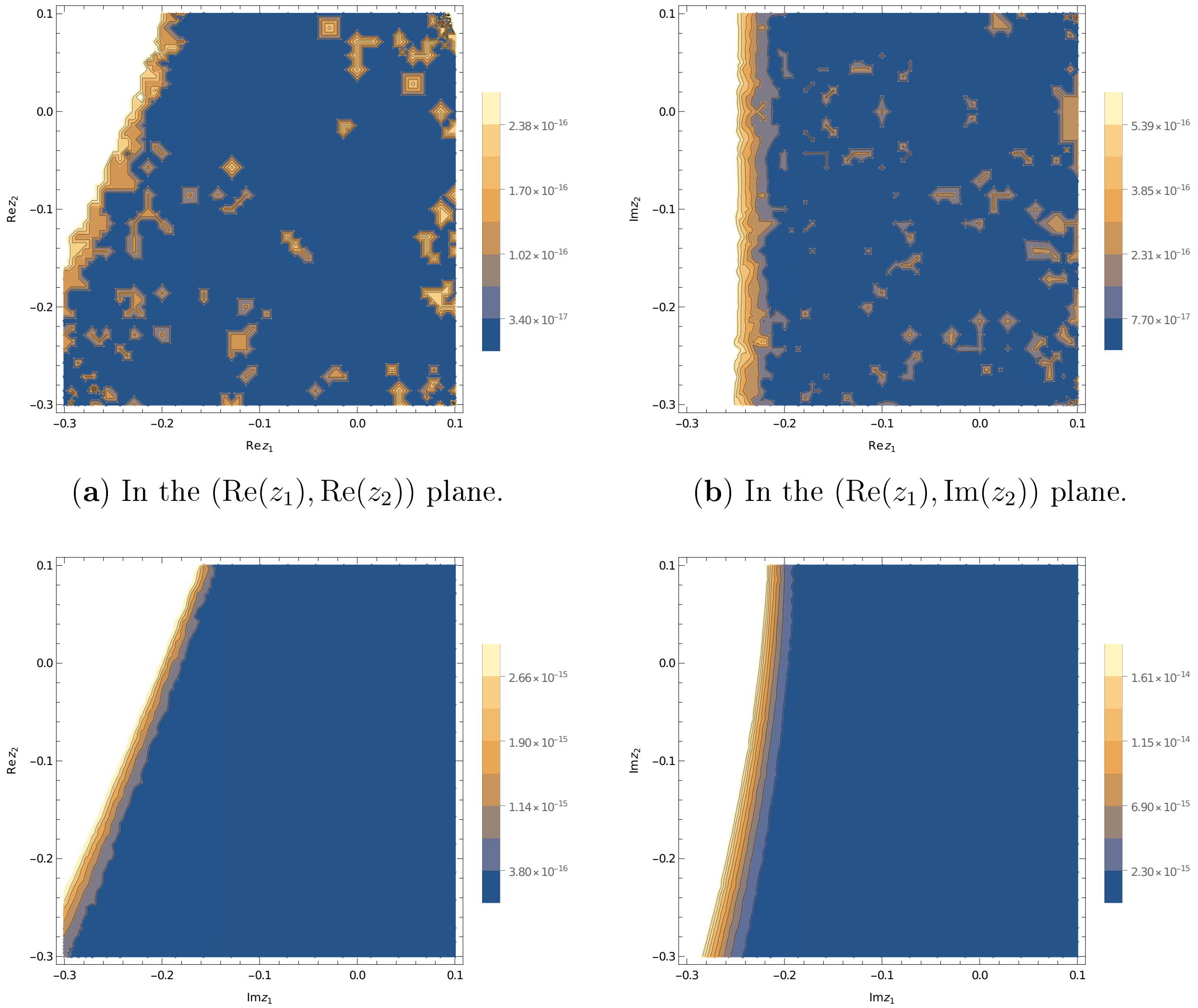
$$\frac{1}{\sqrt{2}}=((1+1/4)^2+4(7/64))^{-1/2}$$

with an error not exceeding 10^{-5} *by the double power series* (4.8)*, one would need to take 57th partial sum, and that* $1/\sqrt{2}$ *can be computed with an error less than* 10^{-5} *by using the 5th approximant of the branched continued fraction* (3.39)*.*

Table 4.1. Relative error of 10th partial sum of the double power series (4.8) and 10th approximants of the branched continued fraction (3.39) for the function (3.40).

$\mathbf{z}$	**(3.40)**	**(4.8)**	**(3.39)**
$(-1/8, 1/10)$	0.873704	6.62333×10^{-8}	1.98945×10^{-10}
$(1/10, -1/16)$	1.17129	7.23624×10^{-8}	2.01913×10^{-10}
$(-1/10, -1/100)$	0.839152	1.01955×10^{-5}	3.27995×10^{-11}
$(-1/10, -1/10)$	1.56174	6.56397×10^{-4}	5.81362×10^{-8}
$(-1/5, -1/5)$	0.668153	6.5287×10^{-1}	1.43181×10^{-9}
$(-1/8, -1)$	0.471405	$2.92301\times10^{+02}$	3.32075×10^{-14}
$(-2, -1/4)$	0.323381	$9.46661\times10^{+08}$	5.42958×10^{-4}
$(-3, -4)$	0.164399	$6.95343\times10^{+12}$	4.77831×10^{-9}
$(-10, -20)$	0.045596	$2.12733\times10^{+19}$	2.98276×10^{-14}
$(-100, -100)$	0.0097124	$8.3222\times10^{+28}$	1.78609×10^{-16}

In Figure 4.2(a)–(d), we can see the plots, where the 20th approximants of the branched continued fraction (3.39) *guarantees certain truncation error bounds for function of two variables* (3.40)*.*



(**a**) In the $(\mathrm{Re}(z_1), \mathrm{Re}(z_2))$ plane.

(**b**) In the $(\mathrm{Re}(z_1), \mathrm{Im}(z_2))$ plane.

(**c**) In the $(\mathrm{Im}(z_1), \mathrm{Re}(z_2))$ plane.

(**d**) In the $(\mathrm{Im}(z_1), \mathrm{Im}(z_2))$ plane.

Figure 4.2. The plots where the approximant $f_{20}(\mathbf{z})$ of branched continued fraction (3.39) guarantees certain truncation error bounds for the function (3.40).

Let us consider another example.

Example 4.3. *From Example 3.3, Theorem 4.1, and Theorem 1.3 the branched continued fraction* (3.41) *converges and represents a single-valued branch of the analytic function* (3.42) *in the domain* $\mathcal{T} \cup \mathrm{Int}(\mathcal{R}_\kappa)$, *where* $\mathcal{P}$ *and* $\mathcal{R}_\kappa$ *are defined by* (3.43) *and* (4.1), *respectively.*

In Table 4.2, we can see that the 10th approximant of the branched continued fraction (3.41) *is eventually a better approximation to the function (3.42)* *than the corresponding 10th partial sum of the double power series*

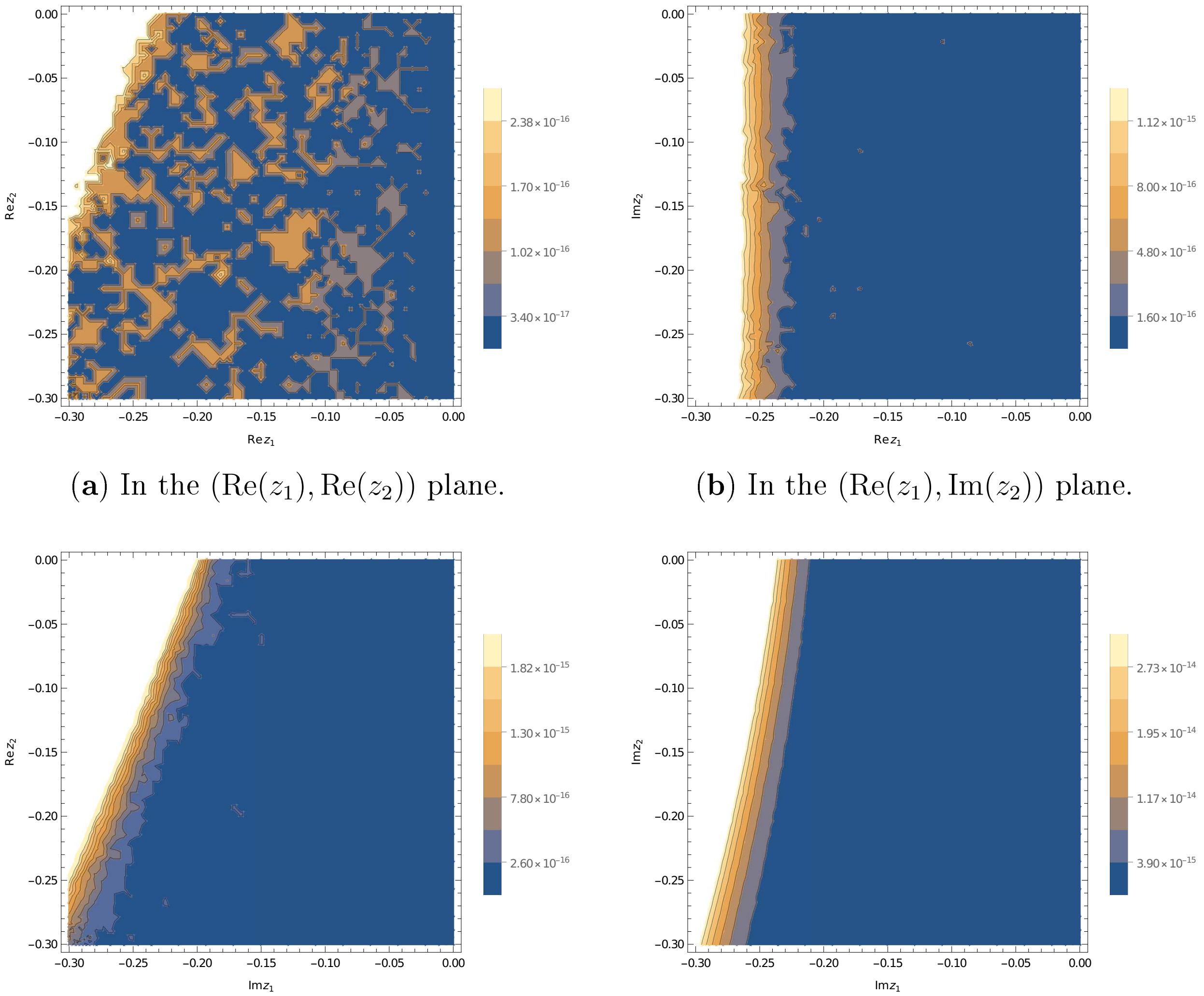
$$\arctan\frac{2\sqrt{-z_1}}{1-z_2} = 2\sqrt{-z_1}H_4(1,b;3/2,b;\mathbf{z}) = \\ = 2\sqrt{-z_1}\sum_{p,q=0}^{\infty}\frac{(1)_{2p+q}}{(3/2)_p}\frac{z_1^p}{p!}\frac{z_2^q}{q!}. \tag{4.9}$$

Table 4.2. Relative error of 10th partial sum of the double power series (4.9) and 10th approximants of the branched continued fraction (3.41) for the function (3.42).

$\mathbf{z}$	**(3.42)**	**(4.9)**	**(3.41)**
$(-1/50, 1/10)$	0.304496	9.2228×10^{-13}	3.6461×10^{-16}
$(-1/8, 1/10)$	0.665944	1.06644×10^{-8}	1.60922×10^{-10}
$(-1/10, -1/100)$	0.559457	2.59086×10^{-6}	2.59205×10^{-11}
$(-1/5, 1/50)$	0.739777	1.46368×10^{-3}	8.30048×10^{-9}
$(-1/5, -1/5)$	0.640522	1.91315×10^{-1}	1.14109×10^{-9}
$(-1/8, -1)$	0.339837	$1.30864 \times 10^{+02}$	2.56454×10^{-14}
$(-4, -1)$	1.10715	$5.94092 \times 10^{+11}$	2.99918×10^{-4}
$(-3, -4)$	0.605891	$2.17485 \times 10^{+12}$	3.78678×10^{-9}
$(-10, -20)$	0.292529	$7.43863 \times 10^{+18}$	2.27715×10^{-14}
$(-100, -100)$	0.195491	$2.65535 \times 10^{+28}$	2.83958×10^{-16}

The graphical illustrations of the function of two variables (3.42) and the branched continued fraction (3.41) are given in Figures 4.3(a)–(d).

Remark 4.1. *The computations in Tables 4.1–4.2 and the plots in Figures 4.1–4.3 were performed using Wolfram Mathematica 13.0 software for Windows 10. Processor Intel(R) Core(TM) i7-4500U CPU 2.40 GHz, Graphics Card Intel(R) HD Graphics Family (113 MB), System Type 64-bit operating system, x64-based processor.*



(**a**) In the $(\mathrm{Re}(z_1), \mathrm{Re}(z_2))$ plane.

(**b**) In the $(\mathrm{Re}(z_1), \mathrm{Im}(z_2))$ plane.

(**c**) In the $(\mathrm{Im}(z_1), \mathrm{Re}(z_2))$ plane.

(**d**) In the $(\mathrm{Im}(z_1), \mathrm{Im}(z_2))$ plane.

Figure 4.3. The plots where the approximant $f_{20}(\mathbf{z})$ of branched continued fraction (3.39) guarantees certain truncation error bounds for the function (3.40).

Next, we will prove the following results.

Theorem 4.5. *Let a, c_1, and c_2 be real constants satisfying the inequalities (3.44), where τ is a positive number, d_0, d_1, h_1 and h_k, $k \geq 2$, are defined in (2.39) and (2.40), respectively. Then:*

(A) The branched continued fraction (2.38) converges to a finite value $f(\mathbf{z})$ for each $\mathbf{z} \in \mathcal{S}_\kappa$, where

$$\mathcal{S}_\kappa = \left\{ \mathbf{z} \in \mathbb{R}^2 : \ z_1 \leq 0, \ z_2 \leq \kappa \right\}, \ 0 < \kappa < \min\left\{ \frac{1}{d_1}, \ 1 \right\}. \qquad (4.10)$$

(B) The convergence is uniform on every compact subset of $\mathrm{Int}(\mathcal{S}_\kappa)$*, and the function* $f(\mathbf{z})$ *is analytic on* $\mathrm{Int}(\mathcal{S}_\kappa)$*.*

(C) For each $\mathbf{z} \in \mathcal{S}_\kappa$ *and for* $n \geq 3$

$$|f(\mathbf{z}) - f_n(\mathbf{z})| \leq$$
$$\leq \frac{|d_0||z_2|(|z_2|(1-z_2)+\kappa|z_1|)(1-z_2)^{-2}(1-z_2+\kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1-d_1z_2)((1-d_1z_2)(1-z_2)+\kappa|z_1|)((1-z_2)^2+\kappa|z_1|)^{n-3}},$$

where $f_n(\mathbf{z})$ *is the nth approximant of the branched continued fraction* (2.38)*.*

(D) The function $f(\mathbf{z})$ *is an analytic continuation of the function (2.25) in the domain* $\mathrm{Int}(\mathcal{S}_\kappa)$*.*

Proof. We will prove (A). Let $\mathbf{z}$ be an arbitrary fixed point in the region (4.10). From the inequalities (3.44) it follows that the coefficients d_1, h_k, $k \geq 1$, are positive real numbers. Using inequalities from (4.10) and the relations (3.25) and (3.27), for any $n \geq 2$ we obtain

$$W_1^{(n)}(\mathbf{z}) = 1 - d_1z_2 - \frac{h_1z_1}{W_2^{(n)}(\mathbf{z})} \geq 1 - d_1z_2 \geq 1 - d_1\kappa > 0,$$

and for arbitraries $n \geq 3$ and $2 \leq k \leq n-1$ we get

$$W_k^{(n)}(\mathbf{z}) = 1 - d_kz_2 - \frac{h_kz_1}{W_{k+1}^{(n)}(\mathbf{z})} \geq 1 - z_2 \geq 1 - \kappa > 0.$$

This allows us to use the formula (1.15). Therefore, for $n \geq 2$ and $k \geq 1$

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = d_0z_1^{n-1}z_2\left(z_2 + \frac{h_nz_1}{W_{n+1}^{(n+k)}(\mathbf{z})}\right)\prod_{r=1}^{n-1}\frac{h_r}{W_r^{(n+k)}(\mathbf{z})W_r^{(n)}(\mathbf{z})}$$

or the same

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = \frac{d_0z_1^{n-1}z_2}{W_1^{(q)}(\mathbf{z})W_n^{(n+k)}(\mathbf{z})}\left(z_2 + \frac{h_nz_1}{W_{n+1}^{(n+k)}(\mathbf{z})}\right)\times$$
$$\times \prod_{r=1}^{[(n-1)/2]}\frac{h_{2r-1}}{W_{2r-1}^{(p)}(\mathbf{z})W_{2r}^{(p)}(\mathbf{z})}\prod_{r=1}^{[(n-2)/2]}\frac{h_{2r}}{W_{2r}^{(q)}(\mathbf{z})W_{2r+1}^{(q)}(\mathbf{z})},$$

where [.] denotes integer part, $q = n+k$, $p = n$, if $n = 2s$, and $q = n$, $p = n+k$, if $n = 2s-1$, $s \geq 1$.

Now, for arbitraries $m \geq 2$ and $k \geq 2$ we have

$$\frac{d_0 z_2}{W_1^{(m)}(\mathbf{z})} \leq \frac{|d_0||z_2|}{1 - d_1 z_2},$$

and

$$\frac{1}{W_m^{(m+k)}(\mathbf{z})}\left(z_2 + \frac{h_m z_1}{W_{m+1}^{(m+k)}(\mathbf{z})}\right) \leq \frac{1}{W_m^{(m+k)}(\mathbf{z})}\left(|z_2| + \frac{h_m |z_1|}{W_{m+1}^{(m+k)}(\mathbf{z})}\right) \leq$$

$$\leq \frac{1}{1 - z_2}\left(|z_2| + \frac{\kappa |z_1|}{1 - z_2}\right).$$

For any $m \geq 2$ we obtain

$$\frac{h_1 z_1}{W_1^{(m+1)}(\mathbf{z}) W_2^{(m+1)}(\mathbf{z})} = \frac{\dfrac{h_1 z_1}{W_2^{(m+1)}(\mathbf{z})}}{1 - d_1 z_2 - \dfrac{h_1 z_1}{W_2^{(m+1)}(\mathbf{z})}} \leq$$

$$\leq \frac{\dfrac{h_1 |z_1|}{W_2^{(m+1)}(\mathbf{z})}}{1 - d_1 z_2 + \dfrac{h_1 |z_1|}{W_2^{(m+1)}(\mathbf{z})}} \leq \frac{\kappa |z_1|}{(1 - d_1 z_2)(1 - z_2) + \kappa |z_1|}.$$

Next, for arbitraries $m \geq 3$ and $2 \leq k \leq m-1$ we get

$$\frac{h_k z_1}{W_k^{(m+1)}(\mathbf{z}) W_{k+1}^{(m+1)}(\mathbf{z})} = \frac{\dfrac{h_k z_1}{W_{k+1}^{(m+1)}(\mathbf{z})}}{1 - z_2 - \dfrac{h_k z_1}{W_{k+1}^{(m+1)}(\mathbf{z})}} \leq \frac{\dfrac{h_k |z_1|}{W_{k+1}^{(m+1)}(\mathbf{z})}}{1 - z_2 + \dfrac{h_k |z_1|}{W_{k+1}^{(m+1)}(\mathbf{z})}} \leq$$

$$le \frac{\kappa |z_1|}{(1 - z_2)^2 + \kappa |z_1|}.$$

And, finally, for any $m \geq 2$ we have

$$\frac{h_m z_1}{W_m^{(m+1)}(\mathbf{z}) W_{m+1}^{(m+1)}(\mathbf{z})} = \frac{h_m z_1}{\left(1 - z_2 - \dfrac{h_m z_1}{W_{m+1}^{(m+1)}(\mathbf{z})}\right) W_{m+1}^{(m+1)}(\mathbf{z})} =$$

$$= \frac{h_m z_1}{1 - z_2 - h_m + z_1} \leq \frac{\kappa |z_1|}{1 - z_2 + \kappa |z_1|}.$$

Thus, for $n \geq 3$ and $k \geq 2$ we obtain

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| \leq$$
$$\leq \frac{|d_0||z_2|(|z_2|(1-z_2)+\kappa|z_1|)(1-z_2)^{-2}(1-z_2+\kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1-d_1z_2)((1-d_1z_2)(1-z_2)+\kappa|z_1|)((1-z_2)^2+\kappa|z_1|)^{n-3}}. \quad (4.11)$$

It is obvious that for an arbitrary fixed $\mathbf{z} \in \mathcal{S}_\kappa$

$$\frac{|d_0||z_2|(|z_2|(1-z_2)+\kappa|z_1|)(1-z_2)^{-2}(1-z_2+\kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1-d_1z_2)((1-d_1z_2)(1-z_2)+\kappa|z_1|)((1-z_2)^2+\kappa|z_1|)^{n-3}} \to 0$$

as $n \to +\infty$. Therefore, by the arbitrariness of k it follows Theorem 4.5(A).

Next, we will prove Theorem 4.5(B). Let $\mathcal{K}$ denotes an arbitrary compact subset of $\mathrm{Int}(\mathcal{S}_\kappa)$. Then there exists an open bi-disk

$$\mathcal{O}_\eta = \{\mathbf{z} \in \mathbb{C}^2 : \ |z_k| < \eta, \ k = 1,2\}$$

of radius η, $\eta > 0$, such that $\mathcal{K} \subset \mathcal{O}_\eta$. Thus for $n \geq 3$, $k \geq 2$ and for all $\mathbf{z} \in \mathcal{K}$ we get

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| <$$
$$< \frac{|d_0|\eta(\eta(1-\kappa)+\kappa\eta)(1-\kappa)^{-2}(1-\kappa+\kappa\eta)^{-2}(\kappa\eta)^{n-1}}{(1-d_1\kappa)((1-d_1\kappa)(1-\kappa)+\kappa\eta)((1-\kappa)^2+\kappa\eta)^{n-3}}$$
$$= \frac{|d_0|(1-\kappa+\kappa)(1-\kappa)^{-2}(1-\kappa+\kappa\eta)^{-2}\kappa^{n-1}\eta^{n+1}}{(1-d_1\kappa)((1-d_1\kappa)(1-\kappa)+\kappa\eta)((1-\kappa)^2+\kappa\eta)^{n-3}}.$$

Next, for arbitrary integer numbers q, p such that $q \geq 2$, $p \geq n \geq 2$, and for all $\mathbf{z} \in \mathcal{K}$, we have

$$|f_{p+q}(\mathbf{z}) - f_p(\mathbf{z})| \leq |f_{p+q}(\mathbf{z}) - f_n(\mathbf{z})| + |f_p(\mathbf{z}) - f_n(\mathbf{z})|.$$

Furthermore, since

$$\frac{|d_0|(1-\kappa+\kappa)(1-\kappa)^{-2}(1-\kappa+\kappa\eta)^{-2}\kappa^{n-1}\eta^{n+1}}{(1-d_1\kappa)((1-d_1\kappa)(1-\kappa)+\kappa\eta)((1-\kappa)^2+\kappa\eta)^{n-3}} \to 0$$

as $n \to +\infty$, it follows Theorem 4.5(B).

Theorem 4.5(C) follows directly from (4.11) if we pass to the limit as $k \to +\infty$.

Next, we will prove Theorem 4.5(D). It is obvious that

$$\frac{H_4(a,c_2+1;c_1,c_2;\mathbf{0})}{H_4(a,c_2+2;c_1,c_2+1;\mathbf{0})}=1.$$

Then, there exists $0<\varepsilon<1$ such that (2.25) is an analytic function in the domain (4.5) and

$$\mathcal{D}_{r,s,\varepsilon}\subset(\mathcal{D}_{r,s}\cap\mathrm{Int}(\mathcal{S}_\kappa)),$$

in particular,

$$\mathcal{D}_{r,s,1/2}\subset(\mathcal{D}_{r,s}\cap\mathrm{Int}(\mathcal{S}_\kappa)),$$

where $\mathcal{D}_{r,s}$ is defined by (4.6), $\mathcal{S}_\kappa$ is defined by (4.11), r and s are positive numbers such that $4r=(s-1)^2$ herewith $s\neq 1$.

Let $\mathbf{z}$ be an arbitrary fixed point in $\mathcal{D}_{r,s,\varepsilon}$. It is clear that all elements of the expansion (2.38) are positive numbers. This means that the approximants of branched continued fraction (2.38) have the property of fork, namely

$$f_{2n}(\mathbf{z})<f_{2n+2}(\mathbf{z})<f_{2n+1}(\mathbf{z})<f_{2n-1}(\mathbf{z}),\ n\geq 1,$$

and, therefore, the sequences $\{f_{2n}(\mathbf{z})\}$ and $\{f_{2n-1}(\mathbf{z})\}$ converge to a finite value $f(\mathbf{z})$.

Let n be an arbitrary natural number. Consider the following

$$\frac{H_4(a,c_2+1;c_1,c_2;\mathbf{z})}{H_4(a,c_2+2;c_1,c_2+1;\mathbf{z})}-f_n(\mathbf{z}),\ n\geq 1,$$

where (see Theorem 2.9)

$$\frac{H_4(a,c_2+1;c_1,c_2;\mathbf{z})}{H_4(a,c_2+2;c_1,c_2+1;\mathbf{z})}=1+\cfrac{d_0z_2}{1-d_1z_2-\cfrac{h_1z_1}{1-\ddots-d_nz_2-\cfrac{h_nz_1}{V_{n+1}^{(n+1)}(\mathbf{z})}}},$$

and

$$V_{n+1}^{(n+1)}(\mathbf{z})=\frac{H_4(a,c_2+n+2;c_1,c_2+n+1;\mathbf{z})}{H_4(a,c_2+n+3;c_1,c_2+n+2;\mathbf{z})}$$

Similar to (3.27) we have

$$V_k^{(n+1)}(\mathbf{z}) = 1 - d_k z_2 - \frac{h_k z_1}{V_{k+1}^{(n+1)}(\mathbf{z})}, \ 1 \le k \le n,$$

where

$$V_k^{(n+1)}(\mathbf{z}) = 1 - d_k z_2 - \cfrac{h_k z_1}{1 - d_{k+1} z_2 - \cfrac{h_{k+1} z_1}{1 - \ddots - d_n z_2 - \cfrac{c_n z_1}{V_{n+1}^{(n+1)}(\mathbf{z})}}}, \ 1 \le k \le n.$$

It is obvious that $W_k^{(n)}(\mathbf{z}) \neq 0$ and $V_k^{(n)}(\mathbf{z}) \neq 0$ for $1 \le k \le n$ and for $\mathbf{z} \in \mathcal{D}_{r,s,\varepsilon}$. Using the formula (1.15), we get

$$\frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})} - f_n(\mathbf{z}) =$$
$$= d_0 z_1^{n-1} z_2 \left(z_2 + \frac{h_{n+1} z_1}{V_{n+1}^{(n+1)}(\mathbf{z})} \right) \prod_{r=1}^{n-1} \frac{c_r}{V_r^{(n+1)}(\mathbf{z}) W_r^{(n)}(\mathbf{z})}.$$

Then, for all $\mathbf{z} \in \mathcal{D}_{r,s,\varepsilon}$ we have

$$f_{2n}(\mathbf{z}) < \frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})} < f_{2n-1}(\mathbf{z}).$$

Next, from the fork property of approximants of branched continued fraction (2.38) it follows that for all $\mathbf{z} \in \mathcal{D}_{r,s,\varepsilon}$

$$\lim_{n \to +\infty} f_{2n}(\mathbf{z}) = \lim_{n \to +\infty} f_{2n-1}(\mathbf{z}) = f(\mathbf{z}),$$

and, therefore, for all $\mathbf{z} \in \mathcal{D}_{r,s,\varepsilon}$,

$$f(\mathbf{z}) = \frac{H_4(a, c_2+1; c_1, c_2; \mathbf{z})}{H_4(a, c_2+2; c_1, c_2+1; \mathbf{z})}.$$

Finally, by Theorem 1.3 and Theorem 4.5(B), Theorem 4.5(D) follows. ■

By Theorem 3.7, Theorem 4.5(B),(D), and Theorem 1.3, we have the following result.

Theorem 4.6. *Let the conditions of Theorem 4.5 be satisfied. Then the branched continued fraction* (2.38) *converges uniformly on every compact subset of the domain*

$$\mathcal{P}_\tau \cup \operatorname{Int}(\mathcal{S}_\kappa),$$

to a function $f(\mathbf{z})$ holomorphic in this domain, and, in addition, the function $f(\mathbf{z})$ is an analytic continuation of the function (2.25) *in this domain, where $\mathcal{P}_\tau$ and $\mathcal{S}_\kappa$ are defined by* (3.45) *and* (4.10), *respectively.*

4.2. Sets of numerical stability

We will establish an estimate of the relative rounding error produced by the backward recurrence algorithm in computing the approximants of the branched continued fraction expansions of Horn hypergeometric series H_4 and their ratios in special cases.

Let n be an arbitrary natural number. To compute the nth approximant

$$f_n = 1 + g_{1,0} + \cfrac{g_{0,1}}{1 + g_{1,1} + \cfrac{g_{0,2}}{1 + \ddots + g_{1,n-2} + \cfrac{g_{0,n-1}}{1 + g_{1,n-1} + g_{0,n}}}}$$

of the branched continued fraction

$$1 + g_{1,0} + \cfrac{g_{0,1}}{1 + g_{1,1} + \cfrac{g_{0,2}}{1 + g_{1,2} + \cfrac{g_{0,3}}{1 + \ddots}}} \tag{4.12}$$

we use the backward recurrence algorithm which consists in setting $G_n^{(n)} = 1$ and computing successively, from tail to head,

$$G_k^{(n)} = 1 + g_{1,k} + \frac{g_{0,k+1}}{G_{k+1}^{(n)}}, \; n-1 \ge k \ge 0,$$

where $g_{1,k-1} \neq 0$, $g_{0,k} \neq 0$, $k \geq 1$. Thus,

$$f_n = G_0^{(n)}.$$

For each $1 \leq k \leq n$, let $\widehat{g}_{1,k-1}$ and $\widehat{g}_{0,k}$ denote rounded values of the elements $g_{1,k-1}$ and $g_{0,k}$, respectively, of a given branched continued fraction (4.12). The number

$$\widehat{f}_n = 1 + \widehat{g}_{1,0} + \cfrac{\widehat{g}_{0,1}}{1 + \widehat{g}_{1,1} + \cfrac{\widehat{g}_{0,2}}{1 + \ddots + \widehat{g}_{1,n-2} + \cfrac{\widehat{g}_{0,n-1}}{1 + \widehat{g}_{1,n-1} + \widehat{g}_{0,n}}}}$$

is the computed (approximate) value of the approximant $f_n = G_0^{(n)}$.

Definition 4.1. *A numerical stability set $\mathcal{D}$ is a set to which for any $\varepsilon > 0$ one can find $\delta > 0$ depending only on ε and $\mathcal{D}$ such that, for all $n \geq 1$,*

$$\left| \frac{\widehat{f}_n - f_n}{f_n} \right| < \varepsilon$$

for every branched continued fraction (4.12) *with all elements $g_{1,k-1}, g_{0,k} \in \mathcal{D}$ and $\widehat{g}_{1,k-1}, \widehat{g}_{0,k} \in \mathcal{D}$ such that, for all $k \geq 1$,*

$$\left| \frac{\widehat{g}_{1,k-1} - g_{1,k-1}}{g_{1,k-1}} \right| < \delta$$

and

$$\left| \frac{\widehat{g}_{0,k} - g_{0,k}}{g_{0,k}} \right| < \delta.$$

Let $\{f_n(\mathbf{z})\}_{n\geq 1}$ and $\{\widehat{f}_n(\mathbf{z})\}_{n\geq 1}$ be the sequences of approximants and their computed values of the branched continued fraction (2.26), respectively. We will find the formula for the relative errors

$$\frac{\widehat{f}_n(\mathbf{z}) - f_n(\mathbf{z})}{f_n(\mathbf{z})}, \ n \geq 1.$$

Again, let n be an arbitrary natural number. For the branched continued fraction (2.26), let α_1, α_2, and β_k, $1 \leq k \leq n$, denote the relative errors in the rounded values $\widehat{z}_1$, $\widehat{z}_2$, and $\widehat{u}_k$, $1 \leq k \leq n$, of z_1, z_2, and u_k, $1 \leq k \leq n$, respectively, so that

$$\widehat{z}_1 = z_1(1 + \alpha_1), \ \widehat{z}_2 = z_2(1 + \alpha_2), \ \widehat{u}_k = u_k(1 + \beta_k), \ 1 \leq k \leq n, \tag{4.13}$$

where u_k, $k \geq 1$, are defined by (2.27). Similarly, let $\varepsilon_k^{(n)}$, $0 \leq k \leq n$, denote the relative errors in $\widehat{F}_k^{(n)}(\widehat{\mathbf{z}})$, the approximation to $F_k^{(n)}(\mathbf{z})$ from (3.11), (3.13) and

$$F_0^{(n)}(\mathbf{z}) = 1 - z_2 - \frac{u_1 z_1}{F_1^{(n)}(\mathbf{z})}$$

using $\widehat{z}_1$, $\widehat{z}_2$, and $\widehat{u}_k$, $1 \leq k \leq n$. Thus,

$$\widehat{F}_k^{(n)}(\widehat{\mathbf{z}}) = F_k^{(n)}(\mathbf{z})(1 + \varepsilon_k^{(n)}),\ 0 \leq k \leq n, \tag{4.14}$$

and

$$\widehat{F}_n^{(n)}(\widehat{\mathbf{z}}) = F_n^{(n)}(\mathbf{z}) = 1,\ \varepsilon_n^{(n)} = 0. \tag{4.15}$$

Also, let $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\beta}_k$, $1 \leq k \leq n$, and $\widehat{\varepsilon}_k^{(n)}$, $0 \leq k \leq n$, denote the relative errors defined by

$$z_1 = \widehat{z}_1(1 + \widehat{\alpha}_1),\ z_2 = \widehat{z}_2(1 + \widehat{\alpha}_2),\ u_k = \widehat{u}_k(1 + \widehat{\beta}_k),\ 1 \leq k \leq n,$$

and

$$F_k^{(n)}(\mathbf{z}) = \widehat{F}_k^{(n)}(\widehat{\mathbf{z}})(1 + \widehat{\varepsilon}_k^{(n)}),\ 0 \leq k \leq n,$$

respectively.

We establish recurrence relations for relative errors $\varepsilon_k^{(n)}$, $0 \leq k \leq n-1$. For arbitrary k, $0 \leq k \leq n-1$, we have

$$\begin{aligned}
\varepsilon_k^{(n)} &= \frac{\widehat{F}_k^{(n)}(\widehat{\mathbf{z}}) - F_k^{(n)}(\mathbf{z})}{F_k^{(n)}(\mathbf{z})} = \frac{1}{F_k^{(n)}(\mathbf{z})}\left(1 - \widehat{z}_2 - \frac{\widehat{u}_{k+1}\widehat{z}_1}{\widehat{F}_{k+1}^{(n)}(\widehat{\mathbf{z}})}\right) - 1 = \\
&= \frac{1}{F_k^{(n)}(\mathbf{z})}\left(1 - z_2(1 + \alpha_2) - \frac{u_{k+1}(1 + \beta_{k+1})z_1(1 + \alpha_1)}{F_{k+1}^{(n)}(\mathbf{z})(1 + \varepsilon_{k+1}^{(n)})}\right) - 1 = \\
&= \frac{1}{F_k^{(n)}(\mathbf{z})} - \frac{z_2(1 + \alpha_2)}{F_k^{(n)}(\mathbf{z})} - \frac{u_{k+1}(1 + \beta_{k+1})z_1(1 + \alpha_1)(1 + \widehat{\varepsilon}_{k+1}^{(n)})}{F_k^{(n)}(\mathbf{z})F_{k+1}^{(n)}(\mathbf{z})} - 1.
\end{aligned}$$

It follows from (3.13) that

$$\frac{1}{F_k^{(n)}(\mathbf{z})} = 1 + \frac{z_2}{F_k^{(n)}(\mathbf{z})} + \frac{u_{k+1}z_1}{F_k^{(n)}(\mathbf{z})F_{k+1}^{(n)}(\mathbf{z})}.$$

Then,

$$\varepsilon_k^{(n)} = \frac{z_2}{F_k^{(n)}(\mathbf{z})} - \frac{z_2(1+\alpha_2)}{F_k^{(n)}(\mathbf{z})} -$$

$$-\frac{u_{k+1}z_1}{F_k^{(n)}(\mathbf{z})F_{k+1}^{(n)}(\mathbf{z})}((1+\beta_{k+1})(1+\alpha_1)(1+\widehat{\varepsilon}_{k+1}^{(n)})-1) =$$

$$= -\frac{z_2\alpha_2}{F_k^{(n)}(\mathbf{z})} - \frac{u_{k+1}z_1}{F_k^{(n)}(\mathbf{z})\widehat{F}_{k+1}^{(n)}(\widehat{\mathbf{z}})}(\beta_{k+1}+\alpha_1+\beta_{k+1}\alpha_1) - \frac{u_{k+1}z_1}{F_k^{(n)}(\mathbf{z})F_{k+1}^{(n)}(\mathbf{z})}\widehat{\varepsilon}_{k+1}^{(n)}.$$

Thus, for each $0 \le k \le n-1$,

$$\varepsilon_k^{(n)} = -\frac{z_2\alpha_2}{F_k^{(n)}(\mathbf{z})} - \frac{u_{k+1}z_1}{F_k^{(n)}(\mathbf{z})\widehat{F}_{k+1}^{(n)}(\widehat{\mathbf{z}})}(\beta_{k+1}+\alpha_1+\beta_{k+1}\alpha_1) - \frac{u_{k+1}z_1}{F_k^{(n)}(\mathbf{z})F_{k+1}^{(n)}(\mathbf{z})}\widehat{\varepsilon}_{k+1}^{(n)}. \quad (4.16)$$

Similarly, for relative errors $\widehat{\varepsilon}_k^{(n)}$, $0 \le k \le n-1$, one obtains

$$\widehat{\varepsilon}_k^{(n)} = -\frac{\widehat{z}_2\widehat{\alpha}_2}{\widehat{F}_k^{(n)}(\mathbf{z})} - \frac{\widehat{u}_{k+1}\widehat{z}_1}{\widehat{F}_k^{(n)}(\widehat{\mathbf{z}})F_{k+1}^{(n)}(\mathbf{z})}(\widehat{\beta}_{k+1}+\widehat{\alpha}_1+\widehat{\beta}_{k+1}\widehat{\alpha}_1) - \frac{\widehat{u}_{k+1}\widehat{z}_1}{\widehat{F}_k^{(n)}(\widehat{\mathbf{z}})\widehat{F}_{k+1}^{(n)}(\widehat{\mathbf{z}})}\varepsilon_{k+1}^{(n)}. \quad (4.17)$$

Combining (4.16) and (4.17) with

$$\rho_k^{(n)}(\mathbf{z}) = \frac{-u_k z_1}{F_{k-1}^{(n)}(\mathbf{z})F_k^{(n)}(\mathbf{z})},\ \widehat{\rho}_k^{(n)}(\widehat{\mathbf{z}}) = \frac{-\widehat{u}_k\widehat{z}_1}{\widehat{F}_{k-1}^{(n)}(\widehat{\mathbf{z}})\widehat{F}_k^{(n)}(\widehat{\mathbf{z}})},\ 1 \le k \le n-1, \quad (4.18)$$

we obtain the following relation

$$\varepsilon_n = \varepsilon_0^{(n)} =$$

$$= \sum_{k=1}^{n} \frac{(-1)^k}{\widetilde{F}_{k-1}^{(n)}} \left(z_{2,k}\alpha_{2,k} + \frac{\widetilde{u}_k z_{1,k}(\widetilde{\beta}_k + \alpha_{1,k} + \widetilde{\beta}_k\alpha_{1,k})}{\widetilde{F}_k^{(n)}} \right) \prod_{r=1}^{k-1} \widetilde{\rho}_r^{(n)}, \quad (4.19)$$

where, for $p = 1, 2$, and $1 \le k \le n-1$,

$$z_{p,k} = \begin{cases} \widehat{z}_p \text{ if } k \text{ even}, \\ z_p \text{ if } k \text{ odd}, \end{cases} \qquad \alpha_{p,k} = \begin{cases} \widehat{\alpha}_p \text{ if } k \text{ even}, \\ \alpha_p \text{ if } k \text{ odd}, \end{cases}$$

and

$$\widetilde{u}_k = \begin{cases} \widehat{u}_k \text{ if } k \text{ even}, \\ u_k \text{ if } k \text{ odd}, \end{cases} \quad \widetilde{\rho}_k^{(n)} = \begin{cases} \rho_k^{(n)}(\mathbf{z}) \text{ if } k \text{ even}, \\ \widehat{\rho}_k^{(n)}(\widehat{\mathbf{z}}) \text{ if } k \text{ odd}, \end{cases}$$

$$\widetilde{F}_k^{(n)} = \begin{cases} F_k^{(n)}(\mathbf{z}) \text{ if } k \text{ even}, \\ \widehat{F}_k^{(n)}(\widehat{\mathbf{z}}) \text{ if } k \text{ odd}. \end{cases}$$

We prove the following theorem.

Theorem 4.7. *Let there exists a constant* α, $0 < \alpha < 1$, *such that*

$$|\alpha_1| \le \alpha, \ |\alpha_2| \le \alpha, \ |\beta_k| \le \alpha \ k \ge 1, \tag{4.20}$$

where α_1, α_2, *and* β_k, $k \ge 1$, *are relative errors of* z_1, z_2, *and* u_k, $k \ge 1$, *respectively, which are defined in* (4.13), $\mathbf{z} \in \mathcal{L}_{\kappa,\tau}$, *where*

$$\mathcal{L}_{\kappa,\tau} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{\kappa(1-\kappa)}{2\tau}, \ |z_2| < \frac{1-\kappa}{2} \right\}, \tag{4.21}$$

and

$$\tau = \sup_{k \ge 1} \{|u_k|, \ |\widehat{u}_k|\}, \ \kappa \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right), \tag{4.22}$$

u_k, $k \ge 1$, *are defined by* (2.27). *Then:*

(A) The set (4.21) *forms the numerical stability set of the branched continued fraction* (2.26).

(B) If ε_n *denotes the relative errors of* n*th approximant of* (2.26), *then*

$$|\varepsilon_n| \le \frac{4\alpha}{(1+\kappa+|3\kappa-1|)(1-\alpha)} \times$$
$$\times \left(\frac{1-\kappa}{2} + \frac{2\kappa(1-\kappa)}{1+\kappa+|3\kappa-1|} \left(2 + \frac{\alpha}{1-\alpha} \right) \right) \frac{1-\varphi(\kappa)^n}{1-\varphi(\kappa)}, \ n \ge 1, \tag{4.23}$$

where

$$\varphi(\kappa) = \begin{cases} \dfrac{2\kappa}{1-\kappa} \ \textit{if} \ 1 < \kappa < \dfrac{1}{3}, \\ \dfrac{1-\kappa}{2\kappa} \ \textit{if} \ \dfrac{1}{3} < \kappa < 1. \end{cases}$$

Proof. We will prove Theorem 4.7(B). To do this, first consider the periodic continued fraction

$$\cfrac{1+\kappa}{2} - \cfrac{\cfrac{\kappa(1-\kappa)}{2}}{\cfrac{1+\kappa}{2} - \cfrac{\cfrac{\kappa(1-\kappa)}{2}}{\cfrac{1+\kappa}{2} - \cfrac{\cfrac{\kappa(1-\kappa)}{2}}{\cfrac{1+\kappa}{2} - \ddots}}}, \tag{4.24}$$

which is equivalent to

$$\cfrac{1+\kappa}{2} - \cfrac{\cfrac{\kappa(1-\kappa)}{1+\kappa}}{1 - \cfrac{\cfrac{2\kappa(1-\kappa)}{(1+\kappa)^2}}{1 - \cfrac{\cfrac{2\kappa(1-\kappa)}{(1+\kappa)^2}}{1 - \ddots}}}, \tag{4.25}$$

since

$$\frac{1+\kappa}{2} \neq 0.$$

Obviously that

$$-\frac{1}{4} < -\frac{2\kappa(1-\kappa)}{(1+\kappa)^2},$$

i.e. the elements of continued fraction (4.25) satisfy the Theorem 1.1. According to this theorem, the continued fraction (4.25) converges, and its value is

$$f^* = \frac{1+\kappa}{2}\left(1 - \frac{1-2\sqrt{1/4 - 2\kappa(1-\kappa)/(1+\kappa)^2}}{2}\right) =$$
$$= \frac{1+\kappa}{2}\left(1 - \frac{1+\kappa-|1-3\kappa|}{2(1+\kappa)}\right) = \frac{1+\kappa+|1-3\kappa|}{4}.$$

The continued fraction (4.24), as equivalent to (4.25), also converges to the value f^*. Moreover, the approximants

$$f_n^* = \frac{1+\kappa}{2} - \cfrac{\frac{\kappa(1-\kappa)}{2}}{\frac{1+\kappa}{2} - \cfrac{\frac{\kappa(1-\kappa)}{2}}{\frac{1+\kappa}{2} - \ddots - \cfrac{\frac{\kappa(1-\kappa)}{2}}{\frac{1+\kappa}{2}}}}, \ n \geq 1,$$

forms a monotonically descending sequence. Indeed, we set

$$F_n^{(n)} = \frac{1+\kappa}{2}, \ n \geq 0,$$

and

$$F_k^{(n)} = \frac{1+\kappa}{2} - \cfrac{\frac{\kappa(1-\kappa)}{2}}{\frac{1+\kappa}{2} - \cfrac{\frac{\kappa(1-\kappa)}{2}}{\frac{1+\kappa}{2} - \ddots - \cfrac{\frac{\kappa(1-\kappa)}{2}}{\frac{1+\kappa}{2}}}}, \ 0 \leq k \leq n-1, \ n \geq 1,$$

where on the right side is a finite continued fraction with $(n-k)$ levels. Then

$$F_k^{(n)} = \frac{1+\kappa}{2} - \frac{\frac{\kappa(1-\kappa)}{2}}{F_{k+1}^{(n)}}, \ 0 \leq k \leq n-1, \ n \geq 1,$$

and

$$f_n^* = \frac{1+\kappa}{2} - \frac{\frac{\kappa(1-\kappa)}{2}}{F_1^{(n)}}, \ n \geq 1.$$

Let n be an arbitrary natural number. By induction on k and (4.22) we show that

$$F_k^{(n)} > \frac{1-\kappa}{2} > 0, \ 1 \leq k \leq n. \tag{4.26}$$

For $k = n$ the inequalities (4.26) are obvious. Let (4.26) hold for $k = r+1$, $r \leq n-1$. Then for $k = r$ we have

$$F_r^{(n)} = \frac{1+\kappa}{2} - \frac{\dfrac{\kappa(1-\kappa)}{2}}{F_{r+1}^{(n)}} >$$

$$> \frac{1+\kappa}{2} - \frac{\dfrac{\kappa(1-\kappa)}{2}}{\dfrac{1-\kappa}{2}} > \frac{1-\kappa}{2} > 0,$$

which proves (4.26).

Now, using (1.15), (4.22), and (4.26) for $n \geq 1$ we have

$$f_{n+1}^* - f_n^* = (-1)^n \frac{(-1)^{n+1}\left(\dfrac{\kappa(1-\kappa)}{2}\right)^{n+1}}{\displaystyle\prod_{r=1}^{n} F_{i(r)}^{(n+1)} F_{i(r)}^{(n)}} < 0,$$

which proves that the sequence $\{f_n^*\}_{n\geq 1}$ is monotonically decreasing.

Again, let n be an arbitrary natural number and $\mathbf{z}$ be an arbitrary fixed point in (4.21). We prove that

$$|F_k^{(n)}(\mathbf{z})| > f_{n-k}^*,\ 0 \leq k \leq n-1, \tag{4.27}$$

where $F_k^{(n)}(\mathbf{z})$, $0 \leq k \leq n-1$, are defined by (3.11) and (3.12).

Using (3.13), (4.21), and (4.22), from (3.13) for $k = n-1$ we have

$$|F_{n-1}^{(n)}(\mathbf{z})| \geq 1 - |z_2| - \frac{|u_n||z_1|}{|F_n^{(n)}(\mathbf{z})|} >$$

$$> 1 - |z_2| - |u_n||z_1| > \frac{1+\kappa}{2} - \frac{\kappa(1-\kappa)}{2} >$$

$$> \frac{1+\kappa}{2} - \frac{\dfrac{\kappa(1-\kappa)}{2}}{\dfrac{1+\kappa}{2}} = f_1^*.$$

Assuming that the inequality (4.27) is true if $k = k+1 \leq n-1$. Then, for

$k = r$ from (3.13) we obtain

$$|F_r^{(n)}(\mathbf{z})| \geq 1 - |z_2| - \frac{|u_{r+1}||z_1|}{|F_{r+1}^{(n)}(\mathbf{z})|} >$$
$$> \frac{1+\kappa}{2} - \frac{\dfrac{\kappa(1-\kappa)}{2}}{f_{n-r-1}^*} = f_{n-r}^*.$$

Since $f_n^* > f^*$ for all $n \geq 1$, then $|F_k^{(n)}(\mathbf{z})| > f^*$ for each $0 \leq k \leq n-1$ and $n \geq 1$. Considering $z_1 \neq 0$, we estimate the terms $\rho_k^{(n)}(\mathbf{z})$, $1 \leq k \leq n-1$, which are defined in (4.18).

For any k, $1 \leq k \leq n-1$, one obtains

$$|\rho_k^{(n)}(\mathbf{z})| = \left|\frac{u_k z_1}{F_{k+1}^{(n)}(\mathbf{z}) F_k^{(n)}(\mathbf{z})}\right| = \left|\frac{\dfrac{u_k z_1}{F_k^{(n)}(\mathbf{z})}}{1 - z_2 - \dfrac{u_k z_1}{F_k^{(n)}(\mathbf{z})}}\right| =$$
$$= \frac{1}{\left|\dfrac{1-z_2}{u_k z_1} F_k^{(n)}(\mathbf{z}) - 1\right|} \leq \frac{1}{\dfrac{1-|z_2|}{|u_k||z_1|}|F_k^{(n)}(\mathbf{z})| - 1} <$$
$$< \frac{1}{\dfrac{(1+\kappa)(1+\kappa+|1-3\kappa|)}{4\kappa(\kappa-1)} - 1} = \begin{cases} \dfrac{2\kappa}{1-\kappa} & \text{if } 1 < \kappa < \dfrac{1}{3}, \\ \dfrac{1-\kappa}{2\kappa} & \text{if } \dfrac{1}{3} < \kappa < 1, \end{cases} = \varphi(\kappa).$$

Now, since $\widehat{\mathbf{z}} = (\widehat{z}_1, \widehat{z}_2) \in \mathcal{L}_{\kappa,\tau}$, then

$$|\widehat{F}_k^{(n)}(\widehat{\mathbf{z}})| > f_{n-k}^*, \ 0 \leq k \leq n-1, \tag{4.28}$$

and

$$\widehat{\rho}_k^{(n)}(\widehat{\mathbf{z}}) < \varphi(\kappa), \ 1 \leq k \leq n-1, \tag{4.29}$$

where $\widehat{F}_k^{(n)}(\widehat{\mathbf{z}})$, $0 \leq k \leq n-1$, and $\widehat{\rho}_k^{(n)}(\widehat{\mathbf{z}})$, $1 \leq k \leq n-1$, are defined in (4.14)–(4.15) and (4.18), respectively. Further, from (4.21) and (4.18) it follows

$$|z_{1,k}| < \frac{\kappa(1-\kappa)}{2\tau}, \ |z_{2,k}| < \frac{1-\kappa}{2}, \ |\widetilde{u}_k| \leq \tau, \ 1 \leq k \leq n.$$

Thus, from (4.19) we have

$$|\varepsilon_n| \leq$$

$$\leq \sum_{k=1}^{n} \frac{1}{|\widetilde{F}_{k-1}^{(n)}|} \left(|z_{2,k}||\alpha_{2,k}| + \frac{|\widetilde{u}_k||z_{1,k}|(|\widetilde{\beta}_k| + |\alpha_{1,k}| + |\widetilde{\beta}_k||\alpha_{1,k}|)}{|\widetilde{F}_k^{(n)}|} \right) \prod_{r=1}^{k-1} |\widetilde{\rho}_r^{(n)}|.$$

Using (4.28)–(4.29), we get

$$|\varepsilon_n| \leq \frac{4\alpha}{(1+\kappa+|3\kappa-1|)(1-\alpha)} \times$$

$$\times \left(\frac{1-\kappa}{2} + \frac{2\kappa(1-\kappa)}{1+\kappa+|3\kappa-1|} \left(2 + \frac{\alpha}{1-\alpha} \right) \right) \sum_{k=1}^{n} \varphi(\kappa)^{k-1},$$

which proves Theorem 4.7(B), since

$$\sum_{k=1}^{n} \varphi(\kappa)^{k-1} = \frac{1-\varphi(\kappa)^n}{1-\varphi(\kappa)}.$$

Finally, we prove Theorem 4.7(A). From (4.23) it follows that there exists a constant C such that

$$|\varepsilon_n| \leq \frac{C\alpha}{1-\alpha},\ n \geq 1.$$

Obviously that, if

$$|\alpha_1| \leq \alpha < \frac{\varepsilon}{\varepsilon + C},\ |\alpha_2| \leq \alpha < \frac{\varepsilon}{\varepsilon + C},$$

and

$$|\beta_k| \leq \alpha < \frac{\varepsilon}{\varepsilon + C},\ k \geq 1,$$

then $|\varepsilon_n| < \varepsilon$, $n \geq 1$, where ε is an arbitrary positive constant. This fact proves that the conditions from Definition 4.1 are fulfilled, which, in turn, proves Theorem 4.7(A) ∎

Now, let $\{f_n(\mathbf{z})\}_{n\geq 1}$ and $\{\widehat{f}_n(\mathbf{z})\}_{n\geq 1}$ be the sequences of approximants and their computed values of the branched continued fraction (2.38), respectively. We will find the formula for the relative errors

$$\frac{\widehat{f}_n(\mathbf{z}) - f_n(\mathbf{z})}{f_n(\mathbf{z})},\ n \geq 1.$$

Let n be an arbitrary natural number, and let α_1, α_2, γ_0, γ_1, and β_k, $1 \le k \le n$, be the relative errors in the rounded values $\widehat{z}_1$, $\widehat{z}_2$, $\widehat{d}_0$, $\widehat{d}_1$, $\widehat{h}_k$, $1 \le k \le n$, of z_1, z_2, d_0, d_1, and h_k, $1 \le k \le n$, respectively, so that

$$\widehat{z}_1 = z_1(1+\alpha_1),\ \widehat{z}_2 = z_2(1+\alpha_2), \tag{4.30}$$

and

$$\widehat{d}_0 = d_0(1+\gamma_0),\ \widehat{d}_1 = d_1(1+\gamma_1),\ \widehat{h}_k = h_k(1+\beta_k),\ 1 \le k \le n, \tag{4.31}$$

where d_0, d_1, h_1 and h_k, $k \ge 2$, are defined in (2.39) and (2.40), respectively. Similarly, for $0 \le k \le n$, let $\varepsilon_k^{(n)}$ be the relative error in $\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})$ of computing $W_k^{(n)}(\mathbf{z})$ using $\widehat{z}_1$, $\widehat{z}_2$, $\widehat{d}_0$, $\widehat{d}_1$, and $\widehat{h}_k$, $1 \le k \le n$. And, therefore,

$$\widehat{W}_k^{(n)}(\widehat{\mathbf{z}}) = W_k^{(n)}(\mathbf{z})(1+\varepsilon_k^{(n)}),\ 0 \le k \le n, \tag{4.32}$$

with initial conditions

$$\widehat{W}_n^{(n)}(\widehat{\mathbf{z}}) = W_n^{(n)}(\mathbf{z}) = 1,\ \varepsilon_n^{(n)} = 0. \tag{4.33}$$

Also, let $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\gamma}_0$, $\widehat{\gamma}_1$, $\widehat{\beta}_k$, $1 \le k \le n$, and $\widehat{\varepsilon}_k^{(n)}$, $0 \le k \le n$, be the relative errors defined by

$$z_1 = \widehat{z}_1(1+\widehat{\alpha}_1),\ z_2 = \widehat{z}_2(1+\widehat{\alpha}_2),\ d_0 = \widehat{d}_0(1+\widehat{\gamma}_0),\ d_1 = \widehat{d}_1(1+\widehat{\gamma}_1),$$

and

$$h_k = \widehat{h}_k(1+\widehat{\beta}_k),\ 1 \le k \le n,\ W_k^{(n)}(\mathbf{z}) = \widehat{W}_k^{(n)}(\widehat{\mathbf{z}})(1+\widehat{\varepsilon}_k^{(n)}),\ 0 \le k \le n,$$

respectively.

Next, we will establish the recurrence relations for $\varepsilon_k^{(n)}$, $0 \le k \le n-1$. For $k = 0$ we have

$$\begin{aligned}\varepsilon_0^{(n)} &= \frac{\widehat{W}_0^{(n)}(\widehat{\mathbf{z}}) - W_0^{(n)}(\mathbf{z})}{W_0^{(n)}(\mathbf{z})} = \frac{1}{W_0^{(n)}(\mathbf{z})}\left(1 + \frac{\widehat{d}_0\widehat{z}_2}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})}\right) - 1 = \\ &= \frac{1}{W_0^{(n)}(\mathbf{z})}\left(1 + \frac{d_0(1+\gamma_0)z_2(1+\alpha_2)}{W_1^{(n)}(\mathbf{z})(1+\varepsilon_1^{(n)})}\right) - 1 = \\ &= \frac{1}{W_0^{(n)}(\mathbf{z})} + \frac{d_0(1+\gamma_0)z_2(1+\alpha_2)(1+\widehat{\varepsilon}_1^{(n)})}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})} - 1.\end{aligned}$$

Since

$$\frac{1}{W_0^{(n)}(\mathbf{z})} = 1 - \frac{d_0 z_2}{W_0^{(n)}(\mathbf{z}) W_1^{(n)}(\mathbf{z})},$$

then

$$\varepsilon_0^{(n)} = \frac{d_0 z_2}{W_0^{(n)}(\mathbf{z}) W_1^{(n)}(\mathbf{z})}((1+\gamma_0)(1+\alpha_2)(1+\widehat{\varepsilon}_1^{(n)}) - 1) =$$
$$= \frac{d_0 z_2}{W_0^{(n)}(\mathbf{z}) \widehat{W}_1^{(n)}(\widehat{\mathbf{z}})}(\gamma_0 + \alpha_2 + \gamma_0\alpha_2) + \frac{d_0 z_2}{W_0^{(n)}(\mathbf{z}) W_1^{(n)}(\mathbf{z})}\widehat{\varepsilon}_1^{(n)}. \tag{4.34}$$

For $k = 1$ we obtain

$$\varepsilon_1^{(n)} = \frac{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}}) - W_1^{(n)}(\mathbf{z})}{W_1^{(n)}(\mathbf{z})} = \frac{1}{W_1^{(n)}(\mathbf{z})}\left(1 - \widehat{d}_1\widehat{z}_2 - \frac{\widehat{h}_1\widehat{z}_1}{\widehat{W}_2^{(n)}(\widehat{\mathbf{z}})}\right) - 1 =$$
$$= \frac{1}{W_1^{(n)}(\mathbf{z})}\left(1 - d_1(1+\gamma_1)z_2(1+\alpha_2) - \frac{h_1(1+\beta_1)z_1(1+\alpha_1)}{W_2^{(n)}(\mathbf{z})(1+\varepsilon_2^{(n)})}\right) - 1 =$$
$$= \frac{1}{W_1^{(n)}(\mathbf{z})} - \frac{d_1(1+\gamma_1)z_2(1+\alpha_2)}{W_1^{(n)}(\mathbf{z})} - \frac{h_1(1+\beta_1)z_1(1+\alpha_1)(1+\widehat{\varepsilon}_2^{(n)})}{W_1^{(n)}(\mathbf{z}) W_2^{(n)}(\mathbf{z})} - 1.$$

Since

$$\frac{1}{W_1^{(n)}(\mathbf{z})} = 1 + \frac{d_1 z_2}{W_1^{(n)}(\mathbf{z})} + \frac{h_1 z_1}{W_1^{(n)}(\mathbf{z}) W_2^{(n)}(\mathbf{z})},$$

then

$$\varepsilon_1^{(n)} = -\frac{d_1 z_2}{W_1^{(n)}(\mathbf{z})}(\gamma_1 + \alpha_2 + \gamma_1\alpha_2) -$$
$$= -\frac{h_1 z_1}{W_1^{(n)}(\mathbf{z}) W_2^{(n)}(\mathbf{z})}((1+\beta_1)(1+\alpha_1)(1+\widehat{\varepsilon}_2^{(n)}) - 1) =$$
$$= -\frac{d_1 z_2(\gamma_1 + \alpha_2 + \gamma_1\alpha_2)}{W_1^{(n)}(\mathbf{z})} - \frac{h_1 z_1(\beta_1 + \alpha_1 + \beta_1\alpha_1)}{W_1^{(n)}(\mathbf{z}) \widehat{W}_2^{(n)}(\widehat{\mathbf{z}})} - -\frac{h_1 z_1 \widehat{\varepsilon}_2^{(n)}}{W_1^{(n)}(\mathbf{z}) W_2^{(n)}(\mathbf{z})}.$$

Similarly,

$$\widehat{\varepsilon}_1^{(n)} = -\frac{\widehat{d}_1\widehat{z}_2(\widehat{\gamma}_1 + \widehat{\alpha}_2 + \widehat{\gamma}_1\widehat{\alpha}_2)}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} - \frac{\widehat{h}_1\widehat{z}_1(\widehat{\beta}_1 + \widehat{\alpha}_1 + \widehat{\beta}_1\widehat{\alpha}_1)}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}}) W_2^{(n)}(\mathbf{z})} -$$
$$-\frac{\widehat{h}_1\widehat{z}_1\varepsilon_2^{(n)}}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}}) \widehat{W}_2^{(n)}(\widehat{\mathbf{z}})}. \tag{4.35}$$

For k, $2 \le k \le n-1$, and for $n \ge 3$, we have

$$\varepsilon_k^{(n)} = \frac{\widehat{W}_k^{(n)}(\widehat{\mathbf{z}}) - W_k^{(n)}(\mathbf{z})}{W_k^{(n)}(\mathbf{z})} = \frac{1}{W_k^{(n)}(\mathbf{z})}\left(1 - \widehat{z}_2 - \frac{\widehat{h}_k \widehat{z}_1}{\widehat{W}_{k+1}^{(n)}(\widehat{\mathbf{z}})}\right) - 1 =$$

$$= \frac{1}{W_k^{(n)}(\mathbf{z})}\left(1 - z_2(1+\alpha_2) - \frac{h_k(1+\beta_k)z_1(1+\alpha_1)}{W_{k+1}^{(n)}(\mathbf{z})(1+\varepsilon_{k+1}^{(n)})}\right) - 1 =$$

$$= \frac{1}{W_k^{(n)}(\mathbf{z})} - \frac{z_2(1+\alpha_2)}{W_k^{(n)}(\mathbf{z})} - \frac{h_k(1+\beta_k)z_1(1+\alpha_1)(1+\widehat{\varepsilon}_{k+1}^{(n)})}{W_k^{(n)}(\mathbf{z})W_{k+1}^{(n)}(\mathbf{z})} - 1.$$

Since

$$\frac{1}{W_k^{(n)}(\mathbf{z})} = 1 + \frac{z_2}{W_k^{(n)}(\mathbf{z})} + \frac{h_k z_1}{W_k^{(n)}(\mathbf{z})W_{k+1}^{(n)}(\mathbf{z})},$$

then

$$\varepsilon_k^{(n)} = \frac{z_2}{W_k^{(n)}(\mathbf{z})} - \frac{z_2(1+\alpha_2)}{W_k^{(n)}(\mathbf{z})} -$$

$$- - \frac{h_k z_1}{W_k^{(n)}(\mathbf{z})W_{k+1}^{(n)}(\mathbf{z})}((1+\beta_k)(1+\alpha_1)(1+\widehat{\varepsilon}_{k+1}^{(n)}) - 1) =$$

$$= -\frac{z_2\alpha_2}{W_k^{(n)}(\mathbf{z})} - \frac{h_k z_1}{W_k^{(n)}(\mathbf{z})\widehat{W}_{k+1}^{(n)}(\widehat{\mathbf{z}})}(\beta_k + \alpha_1 + \beta_k\alpha_1) - \frac{h_k z_1}{W_k^{(n)}(\mathbf{z})W_{k+1}^{(n)}(\mathbf{z})}\widehat{\varepsilon}_{k+1}^{(n)}.$$

Similarly, for $\widehat{\varepsilon}_k^{(n)}$, $2 \le k \le n-1$, and for $n \ge 3$, we obtain

$$\widehat{\varepsilon}_k^{(n)} = -\frac{\widehat{z}_2\widehat{\alpha}_2}{\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})} - \frac{\widehat{h}_k\widehat{z}_1}{\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})W_{k+1}^{(n)}(\mathbf{z})}(\widehat{\beta}_k + \widehat{\alpha}_1 + \widehat{\beta}_k\widehat{\alpha}_1) -$$

$$-\frac{\widehat{h}_k\widehat{z}_1}{\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})\widehat{W}_{k+1}^{(n)}(\widehat{\mathbf{z}})}\varepsilon_{k+1}^{(n)}. \tag{4.36}$$

Now, from (4.34)–(4.36) for $n \ge 3$ we get the following

$$\varepsilon_n = \varepsilon_0^{(n)} = \frac{d_0 z_2(\gamma_0 + \alpha_2 + \gamma_0\alpha_2)}{W_0^{(n)}(\mathbf{z})\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} + \frac{d_0 z_2 \widehat{\varepsilon}_1^{(n)}}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})} =$$

$$= \frac{d_0 z_2(\gamma_0 + \alpha_2 + \gamma_0\alpha_2)}{W_0^{(n)}(\mathbf{z})\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} +$$

$$+ \left(\widehat{d}_1\widehat{z}_2(\widehat{\gamma}_1 + \widehat{\alpha}_2 + \widehat{\gamma}_1\widehat{\alpha}_2) + \frac{\widehat{h}_1\widehat{z}_1(\widehat{\beta}_1 + \widehat{\alpha}_1 + \widehat{\beta}_1\widehat{\alpha}_1)}{W_2^{(n)}(\mathbf{z})}\right)\frac{\pi_0^{(n)}(\mathbf{z})}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} -$$

$$-\pi_0^{(n)}(\mathbf{z})\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})\varepsilon_2^{(n)}$$

and

$$\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})\varepsilon_2^{(n)} = \frac{-\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})}{W_2^{(n)}(\mathbf{z})}\left(z_2\alpha_2 + \frac{h_2 z_1(\beta_2+\alpha_1+\beta_2\alpha_1)}{\widehat{W}_3^{(n)}(\widehat{\mathbf{z}})}\right) - $$

$$-\frac{h_2 z_1}{W_2^{(n)}(\mathbf{z})W_3^{(n)}(\mathbf{z})}\widehat{\pi}_1^{(n)}(\widehat{\mathbf{z}})\widehat{\varepsilon}_3^{(n)} =$$

$$= \sum_{k=2}^{n-1}\frac{(-1)^{k-1}}{\widetilde{W}_k^{(n)}}\left(z_{2,k}\alpha_{2,k} + \frac{\widetilde{h}_k z_{1,k}(\widetilde{\beta}_k+\alpha_{1,k}+\widetilde{\beta}_k\alpha_{1,k})}{\widetilde{W}_{k+1}^{(n)}}\right)\prod_{r=2}^{k-1}\widetilde{\pi}_r^{(n)},$$

where

$$z_{r,k} = \begin{cases} z_r \text{ if } k \text{ even,} \\ \widehat{z}_r \text{ if } k \text{ odd,}\end{cases} \quad \alpha_{r,k} = \begin{cases} \alpha_r \text{ if } k \text{ even,} \\ \widehat{\alpha}_r \text{ if } k \text{ odd,}\end{cases} \quad r = 1, 2,$$

$$\widetilde{\beta}_k = \begin{cases} \beta_k \text{ if } k \text{ even,} \\ \widehat{\beta}_k \text{ if } k \text{ odd,}\end{cases} \quad \widetilde{h}_k = \begin{cases} h_k \text{ if } k \text{ even,} \\ \widehat{h}_k \text{ if } k \text{ odd,}\end{cases}$$

$$\widetilde{\pi}_k^{(n)} = \begin{cases} \pi_k^{(n)}(\mathbf{z}) \text{ if } k \text{ even,} \\ \widehat{\pi}_k^{(n)}(\widehat{\mathbf{z}}) \text{ if } k \text{ odd,}\end{cases} \quad \widetilde{W}_k^{(n)} = \begin{cases} W_k^{(n)}(\mathbf{z}) \text{ if } k \text{ even,} \\ \widehat{W}_k^{(n)}(\widehat{\mathbf{z}}) \text{ if } k \text{ odd,}\end{cases}$$

$$\pi_0^{(n)}(\mathbf{z}) = -\frac{v_0 z_2}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})}, \tag{4.37}$$

and, for $1 \leq k \leq [(n-1)/2]$,

$$\pi_{2k}^{(n)}(\mathbf{z}) = -\frac{h_{2k}z_1}{W_{2k}^{(n)}(\mathbf{z})W_{2k+1}^{(n)}(\mathbf{z})},\ \widehat{\pi}_{2k-1}^{(n)}(\widehat{\mathbf{z}}) = -\frac{\widehat{h}_{2k-1}\widehat{z}_1}{\widehat{W}_{2k-1}^{(n)}(\widehat{\mathbf{z}})\widehat{W}_{2k}^{(n)}(\widehat{\mathbf{z}})}, \tag{4.38}$$

where [.] denotes integer part.

Thus, for $n \geq 3$,

$$\varepsilon_n = \frac{d_0 z_2(\gamma_0+\alpha_2+\gamma_0\alpha_2)}{W_0^{(n)}(\mathbf{z})\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} +$$

$$+\left(\widehat{d}_1\widehat{z}_2(\widehat{\gamma}_1+\widehat{\alpha}_2+\widehat{\gamma}_1\widehat{\alpha}_2) + \frac{\widehat{h}_1\widehat{z}_1(\widehat{\beta}_1+\widehat{\alpha}_1+\widehat{\beta}_1\widehat{\alpha}_1)}{W_2^{(n)}(\mathbf{z})}\right)\frac{\pi_0^{(n)}(\mathbf{z})}{\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})} +$$

$$+\sum_{k=2}^{n-1}\frac{(-1)^k}{\widetilde{W}_k^{(n)}}\left(z_{2,k}\alpha_{2,k} + \frac{\widetilde{h}_k z_{1,k}(\widetilde{\beta}_k+\alpha_{1,k}+\widetilde{\beta}_k\alpha_{1,k})}{\widetilde{W}_{k+1}^{(n)}}\right)\pi_0^{(n)}(\mathbf{z})\prod_{r=1}^{k-1}\widetilde{\pi}_r^{(n)}. \tag{4.39}$$

Finally, we prove the following theorem.

Theorem 4.8. *Let there exists a constant* α, $0 < \alpha < 1$, *such that*

$$|\alpha_1| \le \alpha,\ |\alpha_2| \le \alpha,\ |\gamma_0| \le \alpha,\ |\gamma_1| \le \alpha,\ |\beta_k| \le \alpha,\ k \ge 1, \tag{4.40}$$

where α_1, α_2, γ_0, γ_1, β_k, $k \ge 1$, *are relative errors of* z_1, z_2, d_0, d_1, h_k, $k \ge 1$, *respectively, which are defined in* (4.30) *and* (4.31), $\mathbf{z} \in \mathcal{D}_{\kappa,\tau,\upsilon}$, *where*

$$\mathcal{D}_{\kappa,\tau,\upsilon} = \left\{\mathbf{z} \in \mathbb{C}^2 :\ |z_1| < \frac{\kappa(1-\kappa)}{2\tau},\ |z_2| < \frac{1-\kappa}{2\upsilon}\right\}, \tag{4.41}$$

$$\tau = \max\left\{\sup_{k\in\mathbb{N}} |h_k|,\ \sup_{k\in\mathbb{N}} |\widehat{h}_k|\right\},\ \upsilon = \max\{d_1,\ 1\},\ \kappa \in (0,1) \setminus \{1/3\}, \tag{4.42}$$

d_0, d_1, h_k, $k \ge 1$, *are defined in* (2.39) *and* (2.40), $|d_0| < \upsilon$, *and* $\upsilon_1 \ge 0$. *Then:*

(A) The set (4.41) *forms the numerical stability set of the branched continued fraction* (2.38).

(B) If ε_n *denotes the relative errors of nth approximant of the branched continued fraction* (2.38), *then*

$$\begin{aligned} |\varepsilon_n| &< \frac{2|d_0|(1-\kappa)\alpha}{1+\kappa+|1-3\kappa|}\times \\ &\times\left(\frac{2+\alpha}{\upsilon-|d_0|} + \frac{4(1-\kappa)}{\upsilon(1+\kappa+|1-3\kappa|)-2|d_0|(1-\kappa)}\times\right. \\ &\left.\times\left(\frac{3+\alpha}{2}+\frac{4\kappa(2+\alpha)}{1+\kappa+|1-3\kappa|}\right)\frac{\varphi(\kappa)-\varphi(\kappa)^{n-1}}{1-\varphi(\kappa)}\right),\ n \ge 3, \end{aligned} \tag{4.43}$$

where

$$\varphi(\kappa) = \begin{cases} \dfrac{2\kappa}{1-\kappa} & \text{if } 1 < \kappa < \dfrac{1}{3}, \\ \dfrac{1-\kappa}{2\kappa} & \text{if } \dfrac{1}{3} < \kappa < 1. \end{cases} \tag{4.44}$$

Proof. We will prove Theorem 4.7(B). Let n be an arbitrary natural number that satisfy the inequality $n \ge 3$ and $\mathbf{z}$ be an arbitrary fixed point in the domain (4.41). We will show that

$$|W_k^{(n)}(\mathbf{z})| > f_{n-k},\ 1 \le k \le n-1, \tag{4.45}$$

where $W_k^{(n)}(\mathbf{z})$, $1 \le k \le n-1$, are defined in (3.25) and (3.26), f_k, $1 \le k \le n-1$, are the approximants of the continued fraction (4.24).

From (3.27) for $k = n-1$ we have

$$|W_{n-1}^{(n)}(\mathbf{z})| \ge 1 - v_{n-1}|z_2| - |u_{n-1}||z_1| > \frac{1+\kappa}{2} - \frac{\kappa(1-\kappa)}{2} >$$
$$> \frac{1+\kappa}{2} - \frac{\frac{\kappa(1-\kappa)}{2}}{\frac{1+\kappa}{2}} = f_1.$$

Let (4.45) holds for $k = r+1 \le n-1$. Then, using the induction assumption, for $k = r$ from (3.27) we obtain

$$|W_r^{(n)}(\mathbf{z})| = \left|1 - d_r z_2 - \frac{h_r z_1}{W_{r+1}^{(n)}(\mathbf{z})}\right| \ge 1 - d_r|z_2| - \frac{|h_r||z_1|}{|W_{r+1}^{(n)}(\mathbf{z})|} >$$
$$> \frac{1+\kappa}{2} - \frac{\frac{\kappa(1-\kappa)}{2}}{f_{n-s-1}} = f_{n-s},$$

which proves (4.45).

The sequence $\{f_n\}$ is monotonically descending, as is the sequence of the equivalent continued fraction (4.24). Then $f_n > f$, $n \ge 1$, and, therefore,

$$|W_k^{(n)}(\mathbf{z})| > f, \ 1 \le k \le n-1,$$

where

$$f = \frac{1+\kappa+|1-3\kappa|}{4}.$$

Now, we estimate the following

$$|W_0^{(n)}(\mathbf{z})| = \left|1 + \frac{d_0 z_2}{W_1^{(n)}(\mathbf{z})}\right| \ge 1 - \frac{|d_0||z_2|}{|W_1^{(n)}(\mathbf{z})|} >$$
$$> 1 - \frac{2|d_0|(1-\kappa)}{v(1+\kappa+|1-3\kappa|)} \ge 1 - \frac{|d_0|}{v}. \tag{4.46}$$

Next, we estimate the terms $\pi_0^{(n)}(\mathbf{z})$, $\pi_{2k}^{(n)}(\mathbf{z})$, $1 \le k \le [(n-1)/2]$, which are defined in (4.37) and (4.38). We have

$$|\pi_0^{(n)}(\mathbf{z})| = \left|\frac{d_0 z_2}{W_0^{(n)}(\mathbf{z})W_1^{(n)}(\mathbf{z})}\right| \le \frac{\dfrac{|d_0||z_2|}{|W_1^{(n)}(\mathbf{z})|}}{1-\dfrac{|d_0||z_2|}{|W_1^{(n)}(\mathbf{z})|}} <$$

$$< \frac{2|d_0|(1-\kappa)}{\upsilon(1+\kappa+|1-3\kappa|)-2|d_0|(1-\kappa)}, \tag{4.47}$$

and for any k, $1 \le k \le [(n-1)/2]$, we get

$$|\pi_{2k}^{(n)}(\mathbf{z})| = \left|\frac{h_{2k} z_1}{W_{2k}^{(n)}(\mathbf{z})W_{2k+1}^{(n)}(\mathbf{z})}\right| \le \frac{\dfrac{|h_{2k}||z_1|}{|W_{2k+1}^{(n)}(\mathbf{z})|}}{1-d_{2k}|z_2|-\dfrac{|h_{2k}||z_1|}{|W_{2k+1}^{(n)}(\mathbf{z})|}} <$$

$$< \frac{\kappa(1-\kappa)}{(1+\kappa)\dfrac{1+\kappa+|1-3\kappa|}{4}-\kappa(1-\kappa)} = \varphi(\kappa), \tag{4.48}$$

where $\varphi(\kappa)$ is defined by (4.44). Now, since $\widehat{\mathbf{z}} \in \mathcal{D}_{\kappa,\tau,\upsilon}$, where $\mathcal{D}_{\kappa,\tau,\upsilon}$ is defined by (4.41), then

$$|\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})| > f_{n-k},\ 1 \le k \le n-1, \tag{4.49}$$

and, therefore,

$$\widehat{\pi}_{2k-1}^{(n)}(\widehat{\mathbf{z}}) < \varphi(\kappa),\ 1 \le k \le \left[\frac{n-1}{2}\right], \tag{4.50}$$

where $\widehat{W}_k^{(n)}(\widehat{\mathbf{z}})$, $1 \le k \le n-1$, and $\widehat{\pi}_{2k-1}^{(n)}(\widehat{\mathbf{z}})$, $1 \le k \le [(n-1)/2]$, are defined in (4.33) and (4.38). In addition, from the conditions of this theorem it follows

$$|z_{1,k}| < \frac{\kappa(1-\kappa)}{2\tau},\ |z_{2,k}| < \frac{1-\kappa}{2\upsilon},\ |\widetilde{h}_k| \le \tau,\ 1 \le k \le n.$$

Now, from (4.39) we have

$$|\varepsilon_n| \le \frac{|d_0||z_2|}{|W_0^{(n)}(\mathbf{z})||\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})|}(|\gamma_0|+|\alpha_2|+|\gamma_0||\alpha_2|)+$$

$$+\Bigg(|\widehat{d}_1||\widehat{z}_2|(|\widehat{\gamma}_1|+|\widehat{\alpha}_2|+|\widehat{\gamma}_1||\widehat{\alpha}_2|)+$$

$$+\frac{|\widehat{h}_1||\widehat{z}_1|(|\widehat{\beta}_1|+|\widehat{\alpha}_1|+|\widehat{\beta}_1||\widehat{\alpha}_1|)|}{|W_2^{(n)}(\mathbf{z})|}\Bigg)\frac{|\pi_0^{(n)}(\mathbf{z})|}{|\widehat{W}_1^{(n)}(\widehat{\mathbf{z}})|}+$$

$$+\sum_{k=2}^{n-1}\frac{|\pi_0^{(n)}(\mathbf{z})|}{|\widetilde{W}_k^{(n)}|}\left(|z_{2,k}||\alpha_{2,k}|+\frac{|\widetilde{h}_k||z_{1,k}|(|\widetilde{\beta}_k|+|\alpha_{1,k}|+|\widetilde{\beta}_k|\alpha_{1,k}|)}{|\widetilde{W}_{k+1}^{(n)}|}\right)\prod_{r=1}^{k-1}|\widetilde{\pi}_r^{(n)}|.$$

Using (4.41)–(4.50), we get

$$|\varepsilon_n|<\frac{2|v_0|(1-\kappa)\alpha}{1+\kappa+|1-3\kappa|}\left(\frac{2+\alpha}{v-|v_0|}+\frac{4(1-\kappa)}{v(1+\kappa+|1-3\kappa|)-2|v_0|(1-\kappa)}\times\right.$$

$$\left.\times\left(\frac{3+\alpha}{2}+\frac{4\kappa(2+\alpha)}{1+\kappa+|1-3\kappa|}\right)\sum_{k=2}^{n-1}\varphi(\kappa)^{k-1}\right).$$

Since

$$\sum_{k=2}^{n-1}\varphi(\kappa)^{k-1}=\varphi(\kappa)\frac{1-\varphi(\kappa)^{n-2}}{1-\varphi(\kappa)}$$

then it follows (4.43), which proves Theorem 4.8(B).

Finally, we will prove Theorem 4.8(A). Let $\kappa\in(0,1)\setminus\{1/3\}$. Then we consider the function $\psi(\alpha)$ defined by the right side of (4.43). Since

$$\lim_{\alpha\to 0^+}\psi(\alpha)=0,$$

then for any $\epsilon>0$ there exists $\delta_\epsilon>0$ such that for any $0<\alpha<\delta_\epsilon$

$$\psi(\alpha)<\epsilon.$$

Thus, if

$$|\alpha_1|\le\alpha<\delta_\epsilon,\ |\alpha_2|\le\alpha<\delta_\epsilon,\ |\gamma_0|\le\alpha<\delta_\epsilon,\ |\gamma_1|\le\alpha<\delta_\epsilon,$$

and

$$|\beta_k|\le\alpha<\delta_\epsilon\ k\ge 1,$$

then

$$|\epsilon_n|\le\psi(\alpha)<\epsilon,\ n\ge 3,$$

which together with Definition 4.1 proves Theorem 4.8(A). ■

In this chapter, the estimates of the convergence rate of branched continued fraction expansions of Horn hypergeometric series H_4 and their ratios in special cases are established, as well as, the sets of numerical stability for these expansion. In addition, the estimates of relative rounding errors produced by the backward recurrence algorithm in computing the approximants of the above-mentioned expansion are founded.

The results presented in this chapter were published in [58–60, 62, 70, 71].

CONCLUSIONS

The thesis is devoted to classical problems of establishing recurrence relations of the Horn hypergeometric series H_4, constructing expansions of these series and their ratios in special cases into branched continued fractions, establishing convergence criteria of these expansions and estimates of approximation errors for them, domains of analytical continuation of Horn hypergeometric functions H_4 and their ratios in special cases, and establishing sets of numerical stability of branched continued expansions of these functions.

The following scientific results were obtained in the work:

1. New three- and four-term recurrence relations for the Horn hypergeometric series H_4 are established.

2. Expansions of the Horn hypergeometric series H_4 and their ratios in special cases into branched continued fractions are constructed.

3. Convergence criteria are established for expansions of Horn hypergeometric functions H_4 and their ratios in special cases into branched continued fractions in the cases of real and complex parameters.

4. The domains of analytical continuation for the Horn hypergeometric functions H_4 and their ratios in special cases into branched continued fractions in the cases of real and complex parameters are established in the space $\mathbb{C}^2$;

5. Truncation error bounds are established for expansions of Horn hypergeometric functions H_4 and their ratios in special cases into branched continued fractions in some regions in the space $\mathbb{R}^2$;

6. Sets of numerical stability are established for expansions of ratios of Horn hypergeometric functions H_4 in special cases into branched continued fractions in the space $\mathbb{C}^2$.

The results of the thesis are a contribution to the analytical theory of continued and branched continued fractions. The proposed approaches to studying the convergence and numerical stability of expansions of Horn hypergeometric series H_4 and their ratios in special cases into branched continued fractions can be used to study the convergence of expansions of multiple hypergeometric series and their ratios into branched continued fractions, and the constructed branched continued fraction expansions can be applied to the approximation of analytical functions of two variables that arise in applied problems in mathematics, physics, and engineering.

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APPENDICES

List of publications on the topic of the thesis

1. Дмитришин Р., Луців І.-А., Дмитришин М., Чезарано К. Про деякі області збіжності гіллястих ланцюгових дробових розвинень відношень гіпергеометричних функцій Горна H_4 // Укр. мат. журн. — 2024. — Том. 76, № 4. — С. 502—508 (Engl. transl.: Dmytryshyn R., Lutsiv I.-A., Dmytryshyn M., Cesarano C. On some domains of convergence of branched continued fraction expansions of the ratios of Horn hypergeometric functions H_4 // Ukr. Math. J. — 2024. Vol. 76, № 4. — P. 559–565).
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on Modern Probl. of Anal., Optim., Approx. and Their Appl. (June 25-27, 2025, Rome, Italy): Proc. Book. — Rome, 2025. — P. 59–60.

Information on approval of thesis results

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2. V International Conference dedicated to the 145th anniversary of the birth of Hans Hahn (Chernivtsi, Ukraine, September 23-27, 2024).
3. International Workshop on Current Trends in Analysis and Approximation Theory (Rome, Italy, July 18, 2023)
4. International Online Conference "Current Trends in Abstract and Applied Analysis" (Ivano-Frankivsk, Ukraine, May 12–15, 2022).
5. International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications (Rome, Italy, June 25-27, 2025)
6. Scientific seminar of the Department of Mathematical and Functional Analysis of Vasyl Stefanyk Carpathian National University (Ivano-Frankivsk, November 13, 2024, November 11, 2025, seminar leader: Prof. Dr. A. V. Zagorodnyuk).